

# Optimal $(v, 5, 2, 1)$ optical orthogonal codes with $v \leq 104$

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**Abstract.** Optimal  $(v, 5, 2, 1)$  optical orthogonal codes (OOC) with  $v \leq 104$  are classified up to equivalence.

## 1 Introduction

Since the introduction of fundamental principles of optical code-division multiple-access (OCDMA) using on-off pulses as signature sequences, the search for powerful code structures began. Among the most famous codes introduced to date are optical orthogonal codes (OOCs). They also have applications in mobile radio, frequency-hopping spread-spectrum communications, radar, sonar signal design, constructing protocol-sequence sets for the M-active-out-of T users collision channel without feedback, etc.

OOCs may also be viewed as constant weight error-correcting codes in which any two codewords are cyclically distinct. Also, some balanced incomplete block designs satisfy the requirements of an OOC.

## 2 Preliminaries

For the basic concepts and notations concerning optical orthogonal codes and related designs we follow [3] and [5]. Let us denote by  $Z_v$  the ring of integers modulo  $v$ .

**Definition 2.1** *A  $(v, k, \lambda_a, \lambda_c)$  optical orthogonal code (OOC) can be defined as a collection  $\mathcal{C} = \{C_1, \dots, C_s\}$  of  $k$ -subsets (codeword-sets) of  $Z_v$  such that any two distinct translates of a codeword-set share at most  $\lambda_a$  elements while any two translates of two distinct codeword-sets share at most  $\lambda_c$  elements:*

$$|C_i \cap (C_i + t)| \leq \lambda_a, \quad 1 \leq i \leq s, \quad 1 \leq t \leq v - 1 \quad (1)$$

$$|C_i \cap (C_j + t)| \leq \lambda_c, \quad 1 \leq i < j \leq s, \quad 0 \leq t \leq v - 1. \quad (2)$$

Condition (1) is called the auto-correlation property and (2) the cross-correlation property. The size of  $\mathcal{C}$  is the number  $s$  of its codeword-sets. The larger the size of the code is, the greater its usefulness.

Consider a codeword-set  $C = \{c_1, c_2, \dots, c_k\}$ . Denote by  $\Delta' C$  the multiset of the values of the differences  $c_i - c_j$ ,  $i \neq j$ ,  $i, j = 1, 2, \dots, k$ . The auto-correlation property means that at most  $\lambda_a$  differences are the same. Denote by  $\Delta C$  the underlying set of  $\Delta' C$ . The type of  $C$  is the number of elements of  $\Delta C$ , i.e. the number of different values of its differences. If  $\lambda_c = 1$  the cross-correlation property means that  $\Delta C_1 \cap \Delta C_2 = \emptyset$  for two codeword-sets  $C_1$  and  $C_2$  of the  $(v, k, \lambda_a, 1)$  OOC. A  $(v, k, \lambda_a, 1)$  OOC is *perfect* if  $|\bigcup_{i=1}^s \Delta C_i| = v - 1$ , that is if all nonzero differences are covered.

In [9] it is shown that the size of a  $(v, k, \lambda_a, 1)$ -OOC cannot exceed  $\lambda_a \lfloor \frac{v-1}{k} \rfloor$  but this bound is, in general, far from the tight.  $(v, 4, 2, 1)$ -OOCs have been deeply investigated in [1, 3, 6, 8].

The following Theorem has been proved recently in [4].

**Theorem 2.2** *If  $s$  is the size of a  $(v, 5, 2, 1)$ -OOC, then we have:*

$$s \leq \begin{cases} \left\lfloor \frac{v}{12} \right\rfloor & \text{for } v \equiv 1 \pmod{132} \text{ or } v \equiv 154 \pmod{924} \\ \left\lfloor \frac{v}{12} \right\rfloor & \text{otherwise.} \end{cases}$$

It is natural to say that a  $(v, 5, 2, 1)$ -OOC is *optimal* when its size  $s$  reaches the upper bound given in the Theorem. The authors of [4] present one optimal  $(v, 5, 2, 1)$ -OOC for any length  $v \leq 62$  when it exists. They also give many direct and recursive constructions for infinite classes of optimal  $(v, 5, 2, 1)$ -OOCs.

In our work we extend the results from [4] by constructing new OOCs with  $v > 62$  and classifying all perfect and optimal OOCs with  $v \leq 104$ .

Two  $(v, k, \lambda_a, \lambda_c)$  optical orthogonal codes are *multiplier equivalent* if they can be obtained from one another by an automorphism of  $Z_v$  and replacement of codeword-sets by some of their translates.

### 3 Classification of $(v, 5, 2, 1)$ -OOCs up to multiplier equivalence

We classify the  $(v, 5, 2, 1)$ -OOCs up to multiplier equivalence applying the well-known techniques of back-track search with minimality test on the partial solutions [7, section 7.1.2]. We first arrange all possibilities for codeword-sets with respect to a lexicographic order defined on them.

We assume that  $c_1 < c_2 < c_3 < c_4 < c_5$  for each codeword-set  $C = \{c_1, c_2, c_3, c_4, c_5\}$ . Define a lexicographic order on the codeword-sets implying that:  $C' = \{c'_1, c'_2, c'_3, c'_4, c'_5\}$  is lexicographically smaller than  $C'' = \{c''_1, c''_2, c''_3, c''_4, c''_5\}$  if the type of  $C'$  is smaller than that of  $C''$ , or if the types of the two codewords are the same and  $c'_i = c''_i$  for  $i < a$  and  $c'_a < c''_a$ . If we replace a

codeword-set  $C \in \mathcal{C}$  with a translate  $C + t \in \mathcal{C}$ , we obtain an equivalent OOC. That is why without loss of generality we assume that each codeword-set of the optimal  $(v, 5, 2, 1)$ - OOCs is lexicographically smaller than the codeword-sets of its translates. This means that  $c_1 = 0$  and when we say that  $C_1$  is mapped to  $C_2$  by the permutation  $\varphi$ , we mean that  $C_2$  is the smallest translate of  $\varphi(C_1)$ .

Let  $\varphi_0, \varphi_1, \dots, \varphi_{m-1}$  be the automorphisms of  $Z_v$ , where  $\varphi_0$  is the identity. We construct an array of all sets of 5 elements of  $Z_v$  which might become codeword-set vectors, i.e. which answer the autocorrelation property and are smaller than all their translates. We find them in lexicographic order. To each constructed set we apply the permutations  $\varphi_i, i = 1, 2, \dots, m - 1$ . If some of them maps it to a smaller set, we do not add the current set since it is already somewhere in the array. If we add the current set to the array, we also add after it the  $m - 1$  sets to which it is mapped by  $\varphi_1, \varphi_2, \dots, \varphi_{m-1}$ .

We then apply backtrack search to choose the codeword-sets of the OOC among all these possibilities for them. We use a parallel implementation of the backtrack search on BlueGene/P [2]. The above described ordering of all the possible codeword-sets allows repeated sets in the array, but makes the minimality test of the partial solutions very fast. By the minimality test we check if the current solution can be mapped to a lexicographically smaller one by the automorphisms of  $Z_v$ .

We also apply a type test to the partial solutions. Suppose we have already found  $r$  codeword-sets of the code. Let  $T$  be the type of the  $r$ -th codeword-set, and let  $d$  be the number of distinct differences covered by the  $r$  sets. We only look for optimal codes, i.e. codes with  $s$  codeword-sets. The type of the remaining codeword-sets (of the array we choose them from) is at least as big as that of the  $r$ -th chosen one. That is why  $d + (s - r)T \leq v - 1$ . If this does not hold, we look for the next possibility for the  $r - 1$ -st codeword-set.

In this way we classify the OOCs up to multiplier equivalence.

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