Sixth International Workshop on Optimal Codes and Related Topics
June 16-22, 2009, Varna, Bulgaria *pp. 184-188*

Parallelisms of $PG(3,4)$ with automorphisms of order 7

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Abstract. A spread is a set of lines of $PG(d,q)$, which partition the point set. A parallelism is a partition of the set of lines by spreads. There is a one-to-one correspondence between the parallelisms of $PG(3,4)$ and the resolutions of the 2-(85,5,1) design of its points and lines. We construct 482 non isomorphic parallelisms with automorphisms of order 7.

1 Introduction

For the basic concepts and notations concerning combinatorial designs, projective spaces, spreads and parallelisms, refer, for instance, to [1], [2], [4], [6], [8], or [14].

Let $V = \{P_i\}_{i=1}^{v}$ be a finite set of *points*, and $\mathcal{B} = \{B_j\}_{j=1}^{b}$ a finite collection of k-element subsets of V, called *blocks*. $D = (V, \mathcal{B})$ is a *2-design* with parameters 2-(v,k,λ) if any 2-subset of V is contained in exactly λ blocks of $\mathcal{B}$.

Two designs are *isomorphic* if there exists a one-to-one correspondence between the point and block sets of the first design and the point and block sets of the second design, and if this one-to-one correspondence does not change the incidence. An *automorphism* is an isomorphism of the design to itself, i.e. a permutation of the points which maps blocks into blocks.

A *parallel class* is a partition of the point set by blocks. A *resolution* of the design is a partition of the collection of blocks by parallel classes. Two resolutions are *isomorphic* if there is an automorphism of the design mapping the first one into the second. An *automorphism of a resolution* is an automorphism of the design, which maps parallel classes into parallel classes.

Two resolutions of one and the same design are *mutually orthogonal* if any two parallel classes, one from the first, and the other from the second resolution, have at most one common block.

A *spread* in $PG(d,q)$ is a set of lines which partition the point set. A *parallelism* is a partition of the set of lines by spreads. There can be line spreads and parallelisms if d is odd.

The incidence of the points and t-dimensional subspaces of $PG(d,q)$ defines a 2-design (see for instance [13, 2.35-2.36]), i.e. the points of this design correspond to the points of the projective space, and the blocks to the t-dimensional subspaces. An automorphism of $PG(d,q)$ is a bijective map on the point set that preserves collinearity, i.e. maps lines into lines, and thus t-dimensional subspaces into t-dimensional subspaces. Therefore all related designs have the full automorphism group of the projective space. There is a one-to-one correspondence between parallelisms and the resolutions of the related point-line design. Isomorphism, automorphisms and orthogonality of parallelisms are defined as for resolutions. A parallelism is called transitive if it has an automorphism group, which is transitive on the spreads.

Parallelisms of $PG(3,4)$ can be obtained by Beutelspacher's general construction of parallelisms in $PG(2^n-1,q)$ [3] , and a pair of orthogonal ones by Fuji-Hara's construction for $PG(3,q)$ [7]. All parallelisms of $PG(3,2)$ are known. Parallelisms with predefined automorphism groups have been classified by Prince in $PG(3,3)$ [11] and $PG(3,5)$ [12]. Before the present work $q=4$ was the smallest q, for which no automorphism classification of parallelisms was done. One of the reasons is the nonexistence of transitive parallelisms [5], which are the easiest case to classify.

We construct parallelisms of $PG(3,4)$ with automorphisms of order 7 and establish that up to equivalence their number is 482.

Our programmes performing the computer computations, are based on the exhaustive back track search techniques (see for instance [9, chapter 4]).To filter away isomorphic parallelisms, we find the normalizer of the subgroup of order 7 in the automorphism group of the projective space.

We use design approach to the problems. We actually make all the computations on the related to PG(3,4) designs, namely, we choose the 17 spread elements among the 357 blocks of the 2-(85,5,1) point-line design, and construct the parallelisms as its resolutions. We find a generating set of the automorphism group of PG(3,4), and a subgroup of order 7 and its normalizer as automorphism groups of the related 2-(85,21,5) point-hyperplane design.

2 Construction and results

There are 85 points and 357 lines in $PG(3,4)$. The full automorphism group of $PG(3,4)$ is of order 1974067200. A spread has 17 lines which partition the point set and a parallelism has 21 spreads.

To construct $PG(3,4)$ we use $GF(4)$ with generating polynomial $x^2=x+1$. The points of $PG(3,4)$ are then all 4-dimensional vectors (v_1,v_2,v_3,v_4) over $GF(4)$ such that if $v_k=0$ for all $k>i$ then $v_i=1$. We sort these 85 vectors in ascending lexicographic order and then assign them numbers such that (1,0,0,0) is number 1, and (3,3,3,1) number 85. We then construct the related designs and find the generators of their full automorphism group G.

Since 7 divides the order of G, but 7^2 does not, by Silow's Theorem 2 (see, for instance[10, 7.2.4]) all subgroups of order 7 are conjugate, and we can choose an arbitrary one of them. Denote it G_7. It is cyclic and fixes one point, while the other 84 points are in 12 orbits of length 7 (Table 1). G_7 partitions the lines into 51 orbits of length 7.

We sort the 357 lines (blocks of the 2-(85,5,1) design) in lexicographic order defined on the numbers of the points they contain and assign to each line a number according to this order. Then the first point is in the first 21 lines. We begin with a construction of all spreads, which contain each of these 21 lines, and for which all spread lines are from different orbits of G_7. For that purpose we perform a backtrack search. If there are already n elements in the spread, we choose the $n + 1$-st one among the lines containing the first point, which is in none of the n spread elements. The spread elements are lexicographically ordered, and any spread we construct is lexicographically greater than the ones constructed before it.

For each spread we already know the other 6 spreads of its orbit under G_7. We call the first spread *orbit leader*. To obtain a parallelism we need 3 orbit leaders which cover all orbits. Without loss of generality we assume that the first orbit leader contains the first line. We choose the next spread such that it begins with the first line from the first non used orbit. It follows from the construction above that the three orbit leaders are ordered lexicographically, and that the orbit leaders sequence of the current parallelism is lexicographically greater than the sequences constructed before it.

This way we construct 26028 parallelisms. Our next task is to filter away isomorphic ones. Let $\varphi \in G$. Let $\mathcal{P}_1$ be a parallelism with automorphism group $G_{\mathcal{P}_1}$, and let $\mathcal{P}_2 = \varphi\mathcal{P}_1$. Denote by $G_{\mathcal{P}_2}$ the automorphism group of the parallelism $\mathcal{P}_2$. Let $\alpha \in G_{\mathcal{P}_1}$ and $\beta \in G_{\mathcal{P}_2}$. Then $\beta\varphi\mathcal{P}_1 = \varphi\alpha\mathcal{P}_1$ and thus $\beta = \varphi\alpha\varphi^{-1}$ and $G_{\mathcal{P}_2} = \varphi G_{\mathcal{P}_1}\varphi^{-1}$. In our case $G_{\mathcal{P}_2} = G_{\mathcal{P}_1} = G_7$ and therefore we are interested in the normalizer $N(G_7)$ of G_7 in G, which is defined as $N(G_7) = \{g \in G \mid gG_7g^{-1} = G_7\}$. If an automorphism $\varphi \in G$ transforms one of the constructed parallelisms into another one, then $\varphi \in N(G_7)$.

The normalizer $N(G_7)$ is a group of order 378. Let $G_{54} = N(G_7)\backslash G_7$ (Table 1). For each parallelism we obtain, we check if an automorphism of G_{54} transforms it into a parallelism with a lexicographically smaller orbit leader sequence, and drop it if so. This way 482 non isomorphic parallelisms remain, i.e. the 26028 parallelisms are in 482 orbits of length 54 under G_{54}. We also established that there are no pairs of orthogonal ones among them.

Acknowledgment. This research was partially supported by the Bulgarian NSF under Contract MM-1405/04.

Table 1:

Generator of G_7:
(1,30,23,31,5,2,22)(3,26,24,33,37,27,29)(4,34,25,32,28,35,36) (6)
(7,46,39,47,15,14,38)(8,78,71,79,20,18,70)(9,62,55,63,13,10,54)
(11,74,40,81,69,59,45)(12,50,73,48,60,67,84)(16,58,72,65,53,43,77)
(17,82,57,80,44,51,68)(19,66,41,64,76,83,52)(21,42,56,49,85,75,61)

Generators of G_{54}:

(1,3,23,37,31,29)(2,33,30,24,22,27)(4,36,28)(5,26)(6)
(7,21,55,69,79,77)(8,16,39,85,63,45)(9,11,71,53,47,61)(10,81,78,72,38,75)
(12,84,60)(13,74,20,58,15,42)(14,49,62,40,70,43)(17,52,44,19,68,76)
(18,65,46,56,54,59)(25,32,35)(34)(41,80,83,57,64,51)(48,67,73)(50)(66,82)

(1,4,29)(2,32,27)(3,23,28)(5,34,26)(6)(7,12,61,8,17,45,9,19,77)
(10,64,43,14,48,75,18,80,59)(11,55,76,16,39,60,21,71,44)(13,66,58,15,50,42,20,82,74)
(22,35,24)(25,33,30)(31,36,37)(38,67,56,70,51,40,54,83,72)
(41,65,46,73,49,78,57,81,62)(47,84,85,79,68,69,63,52,53)

(1)(2)(3,4)(5)(6)(7)(8,9)(10,18)(11,19)(12,21)(13,20)(14)(15)(16,17)
(22)(23)(24,25)(26,34)(27,35)(28,37)(29,36)(30)(31)(32,33)(38)(39)(40,41)
(42,50)(43,51)(44,53)(45,52)(46)(47)(48,49)(54,70)(55,71)(56,73)(57,72)
(58,82)(59,83)(60,85)(61,84)(62,78)(63,79)(64,81)(65,80)(66,74)(67,75)(68,77)(69,76)

(1)(2)(3)(4)(5)(6)(7,8,9)(10,14,18)(11,16,21)(12,17,19)(13,15,20)(22)(23)
(24)(25)(26)(27)(28)(29)(30)(31)(32)(33)(34)(35)(36)(37)(38,70,54)(39,71,55)
(40,72,56)(41,73,57)(42,74,58)(43,75,59)(44,76,60)(45,77,61)(46,78,62)(47,79,63)
(48,80,64)(49,81,65)(50,82,66)(51,83,67)(52,84,68)(53,85,69)

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