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CLASSIFICATION OF THE 2-SPREADS OF $PG(5,2)$*

Svetlana Todorova Topalova, Stela Dimitrova Zhelezova

The 2-spreads of $PG(5,2)$ are constructed and tested for equivalence by a method considering the specific properties of the automorphism group of the projective space. It is established that up to the equivalence there exists only one 2-spread of $PG(5,2)$. The order of the automorphism group preserving it is 10584.

1. Introduction. For the basic concepts and notations concerning combinatorial designs and projective spaces, the reader is referred, for instance, to [1], [2], [3], [5], [6], or [10].

1.1 Projective spaces and spreads. A *projective space* is a geometry consisting of a set of *points* and a set of *lines*, where each line is a subset of the point set, such that the following axioms hold:

- Any two points are on exactly one line.
- Let A, B, C, D be four distinct points no three of which are collinear. If the lines AB and CD intersect each other, then the lines AD and BC also intersect each other.
- Any line has at least 3 points.

Let V be a vector space of dimension $d+1$ over the division ring F. The geometry $P(V)$ that has as its points the 1-dimensional subspaces of V and as its lines the 2-dimensional subspaces of V, is a projective space. Any projective space that is not a projective plane is isomorphic to some $P(V)$, which is also denoted by $PG(d,F)$. If F is a finite field with q elements, then the notation $PG(d,q)$ is used, where d is called dimension, and q order of the projective space, and any line has $q+1$ points.

An automorphism of $PG(d,q)$ is a bijective map of the point set that preserves collinearity, i.e. maps the lines into lines.

A linear subspace of a projective space is a set U of points such that if $A, B \in U$, then any point on the line AB is contained in U. Any subspace together with the lines contained in it, is a projective space. For two lines A and B of $PG(d,q)$, denote by $\langle A, B\rangle$ the subspace of smallest dimension containing them.

A *t-spread* in $PG(d,q)$ is a set S of t-dimensional subspaces such that any point of the geometry is on exactly one element of S.

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A partial t-spread in $PG(d, q)$ is a set of disjoint t-dimensional subspaces. A partial t-spread in $PG(d, q)$ is maximal if it is not properly contained in any partial t-spread of $PG(d, q)$. In this context a partial t-spread in $PG(d, q)$ which forms a partition of the points, is called a t-spread.

Two partial t-spreads in $PG(d, q)$ are equivalent if there is an automorphism of the geometry, mapping one to the other.

1.2. Combinatorial designs. Let $V = \{P_i\}_{i=1}^{v}$ be a finite set of *points*, and $\mathcal{B} = \{B_j\}_{j=1}^{b}$ a finite collection of k-element subsets of V, called *blocks*. $D = (V, \mathcal{B})$ is a *design* with parameters t-(v,k,λ) if any t-subset of V is contained in exactly λ blocks of $\mathcal{B}$.

Two designs are *isomorphic* if there exists a one-to-one correspondence between the point and block sets of the first design and the point and block sets of the second design, and if this one-to-one correspondence does not change the incidence.

An *automorphism* is an isomorphism of the design onto itself. The set of all automorphisms of a design forms a group called its *full group of automorphisms*. Each subgroup of this group is a group of automorphisms of the design.

A 2-($4m-1$,$2m-1$,$m-1$) design is called a Hadamard 2-design. A 2-(v,3,1) design is called a Steiner triple system of order v ($STS(v)$).

1.3. $PG(5, 2)$–automorphism group, subspaces and related designs. There are 63 points and 651 lines in $PG(5, 2)$. A g-dimensional subspace has $2^{g+1} - 1$ points and $(2^{2g+1} + 1)/3 - 2^g$ lines.

All the 651 lines form an $STS(63)$. The 2-dimensional subspaces of $PG(5, 2)$ form a 2-(63,7,15) design, the 3-dimensional subspaces form a 2-(63,15,35) design and the 4-dimensional subspaces form a Hadamard 2-(63,31,15) design.

Any two intersecting lines of $PG(5, 2)$ determine a 2-dimensional subspace.

The full automorphism group of $PG(5, 2)$ is the projective general linear group $PGL(6, 2)$. This group is doubly transitive both on the points and on the lines. An automorphism fixes either 0 or 1 point, or all the points of a subspace. For each subspace P there is a subgroup fixing its points, and this subgroup is transitive on the points outside P. Thus, the order of the full automorphism group is

$$|PGL(6, 2)| = 63(63 - 1)(63 - 3)(63 - 7)(63 - 15)(63 - 31) = 20158709760.$$

1.4. Known classifications of spreads in $PG(d, q)$. Soicher [9] classified partial 1-spreads of lines in finite projective spaces using the GRAPE package within the GAP system. His results include the construction up to equivalence of all partial 1-spreads in $PG(3, 2)$, $PG(4, 2)$ and $PG(3, 3)$, all maximal partial 1-spreads in $PG(3, 4)$ and the maximal partial 1-spreads in $PG(3, 7)$ of size 45 and invariant under a group of order 5. Then, Blokhuis, Brouwer and Wilbrink classified all maximal partial 1-spreads of size 45 in $PG(3, 7)$ [4].

Looking for affine 2-(64,16,5) designs of small rank, Mavron, McDonough and Tonchev constructed [8] more than 30000 1-spreads of $PG(5, 2)$. Recently, Mateva and Topalova [7] classified, up to equivalence, all 1-spreads of $PG(5, 2)$.

1.5. The present work. The subject of this paper are the 2-spreads of $PG(5, 2)$ whose nine elements are disjoint 2-dimensional subspaces, each of which has 7 points. We use design approach to the problem. We actually make all the computations on the related to PG(5,2) designs, namely, we choose the nine spread elements among the 1395 blocks of the 2-(63,7,15) design, and compute the needed automorphism groups as automorphism groups of the Hadamard 2-(63,31,15) design. We use the specific properties of $PGL(6,2)$, and of the subspaces of $PG(5,2)$ to reduce the search space, to filter away the equivalent spreads, and to establish the uniqueness of the 2-spread.

2. Construction of all inequivalent spreads. Any two intersecting lines A and B of $PG(5,2)$ determine a 2-dimensional subspace $\langle A,B\rangle$. Denote a line through the points a,b and c by $\{a,b,c\}$. Without loss of generality we can denote the points of $PG(5,2)$ by the numbers $1,2,\ldots,63$ in such a way that $\{1,2,3,4,5,6,7\} = \langle\{1,2,3\},\{1,4,5\}\rangle = P_1$ and $\{8,16,17,32,33,34,35\} = \langle\{8,16,17\},\{8,32,33\}\rangle = P_2$ are 2-dimensional subspaces, $\{1,2,\ldots,15\}$ is a 3-dimensional subspace, and $\{1,2,\ldots,31\}$ is a 4-dimensional subspace. In our notations, the related Hadamard 2-(63,31,15) design is presented in Table 1. It contains all the information about the geometry, namely 3 points are on a line if they are together in 15 blocks of this design, 7 points form a 2-dimensional subspace if they are together in 7 blocks, 15 points form a 3-dimensional subspace if they are together in 3 blocks, and the 31 points of each block form a 4-dimensional subspace.

The automorphism group G of the projective space is doubly transitive on the lines, and, thus, each pair of intersecting spread lines can be mapped by some automorphism into the pair $\{1,2,3\}$ and $\{1,4,5\}$. Respectively, each 2-dimensional subspace can be mapped into the space P_1.

For each pair of intersecting lines X and Y, which have no point of P_1, there is an automorphism fixing the points of P_1, and mapping X into $\{8,16,17\}$, and Y into $\{8,32,33\}$. Therefore, for each 2-dimensional subspace P containing no points of P_1, there is an automorphism fixing the points of P_1, and mapping P into the 2-dimensional subspace P_2.

Thus, for each spread, there exists an automorphism mapping it into a spread containing the two elements P_1 and P_2. That is why we only construct spreads containing these two elements.

We realize a backtrack search. If there are already n elements in the spread, we choose the $(n+1)$-st one among the 2-dimensional subspaces containing the first point, which is in none of the n spread elements. We arrange the 2-dimensional subspaces in lexicographic order defined on the numbers of the points they contain. Thus, the spread elements are lexicographically ordered, and any spread we construct is lexicographically greater than the ones constructed before it.

The main problem here is the very big number of isomorphic solutions. Obtaining a spread, we have to check for the existence of an automorphism mapping it into an already constructed one, i.e. into a lexicographically smaller one. However, $PG(5,2)$ has 20158709760 automorphisms, and put in otder all of them makes the computation time impossible.

The number of automorphisms which can map the constructed spreads into one another, is actually smaller, because the first two elements in them are the same.

Let S_1 and S_2 be two equivalent spreads, and let $\alpha \in G$ map the elements of S_2 into the elements of S_1. Let $\alpha Q = P_1$ and $\alpha R = P_2$, where $Q,R \in S_2$. Suppose there exists $\beta \in G$, such that $\beta Q = P_1$ and $\beta R = P_2$.

There exists $\varphi \in G$, such that $\alpha = \varphi\beta$. Then, $\varphi = \alpha\beta^{-1}$. Therefore, φ fixes P_1 and P_2.

Table 1. The 2-(63,31,15) design of the hyperplanes of $PG(5,2)$

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001110011101000010111110001000010111110010000101111100100001011
001000011110101011100010010101011100010101010111000101010101110
010110011100111000101101000111000101101001110001011010011100010
100111111110000101101100110000101101100100001011011001000010110
111001111100010110100011100010110100011000101101000110001011010
000100011101011111000100001011111000100010111110001000101111100
110011111100001011011001100001011011001000010110110010000101101
000010011110110110011000010110110011000101101100110001011011001
101011111101110001001010101110001001010011100010010100111000100
101101111110011000010110110011000010110100110000101101001100001
110101111111100010000101111100010000101111000100001011110001000
011010011111001100001011011001100001011110011000010111100110000
011100011100100101010111000100101010111001001010101110010010101
100001010010010011001111010010011001111011011001100000110110011
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010000110000101110001110100101110001110110100011100011101000111
001110110010111101000001110111101000001010000101111100100001011
111111010000000000000000000000000000000011111111111111111111111
000100110010100000111011110100000111011010111110001000101111100
110101010000011101111010000011101111010111000100001011110001000
101011010010001110110101010001110110101011100010010100111000100
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001000110001010100011101101010100011101101010111000101010101110
100111010001111010010011001111010010011100001011011001000010110
111001010011101001011100011101001011100000101101000110001011010
010110110011000111010010111000111010010001110001011010011100010
100000101010010011001111001101100110000100100110011110110110011
011011001000110011110100111001100001011001100111101001100110000
111110000111111111111111100000000000000000000000000001111111111
000101100101011111000100010100000111011101000001110110101111100
010001100111010001110001000101110001110001011100011101101000111
111110101000000000000000011111111111111000000000000001111111111
010001001000101110001110111010001110001001011100011101101000111
100000000101101100110000110010011001111100100110011110110110011
001111001010111101000001101000010111110101111010000010100001011
001111100101000010111110010111101000001101111010000010100001011
001001001001010100011101110101011100010010101000111011010101110
110010000100000101101100111111010010011011110100100110000010110
010111001011000111010010100111000101101110001110100100011100010
011101100100100101010111011011010101000110110101010000010010101
000011100110110110011000001001001100111010010011001111011011001
100110101001111010010011010000101101100011110100100111000010110
101100000110011000010110101100111101001011001111010011001100001
111000101011101001011100000001011010001111101001011100000101101
100110000110000101101100101111010010011011110100100111000010110
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010111100100111000101101011000111010010110001110100100011100010
001001100110101011100010001010100011101010101000111011010101110
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111000000100010110100011111101001011100111010010111000001011010
101010000010111000100101011000111011010110001110110101011100010
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101100101001100111101001010011000010110011001111010011001100001
110100000111100010000101100011101111010000111011110101110001000
000011001001001001100111101101100110000100100110011110110110011
110010101011110100100110000001011011001111101001001100000101101
011011100111001100001011000110011110100001100111101001100110000

Before starting the spread construction, for each 2-dimensional subspace of $PG(5,2)$ we find and save an automorphism that maps it into P_1. For each disjoint with P_1 2-dimensional subspace we find and save an automorphism that fixes P_1 and maps it into P_2.

When we obtain a new spread, we check if one of the automorphisms $\varphi_n\delta_j\gamma_i$ maps it into a lexicographically smaller one. Here γ_i is an automorphism mapping the spread element P_i into P_1, δ_j is an automorphism fixing P_1 and mapping the 2-dimensional subspace $\gamma_i Q_j$ into P_2, and φ_n is an automorphism fixing the 2-dimensional subspaces P_1 and P_2. Here $i = 1, 2, \dots, 9$, $j = 1, 2, \dots, 8$. The automorphism group preserving P_1 and P_2 is of order 28224, and, thus, $n = 1, 2, \dots, 28224$.

This way instead of trying all the 20158709760 automorphisms, to check for equivalence we use $2032128 = 9.8.28224$ of them, and the computation becomes possible.

3. Results. We obtain 192 2-spreads, containing P_1 and P_2 as two of their nine elements. For each two of them, there is an automorphism among those 2032128 described above, which maps them into one another. So they are all equivalent to the following 2-spread:

$\{1,2,3,4,5,6,7\}$, $\{8,16,17,32,33,34,35\}$, $\{9,18,24,36,37,45,62\}$,
$\{10,19,27,44,50,52,63\}$ $\{11,25,31,38,48,51,56\}$,$\{12,23,30,39,43,47,55\}$,
$\{13,20,22,42,46,54,61\}$ $\{14,26,28,41,49,57,58\}$ $\{15,21,29,40,53,59,60\}$.

We next find out the subgroup of $PGL(6,2)$ which maps the spread into itself. It is of order 10584.

4. Correctness of the results. The construction and classification of the spreads, and the computation of the automorphism groups was done by our own C++ programs, some of them written, and others modified for this specific task. Since programming mistakes are always possible, we obtained by two different programmes the number (192) of all 2-spreads, containing the elements P_1 and P_2, and the order (28224) of the automorphism group fixing P_1 and P_2.

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Svetlana Todorova Topalova
Institute of Mathematics
Bulgarian Academy of Sciences
P.O. Box 323
5000 Veliko Tarnovo, Bulgaria
e-mail: svetlana@moi.math.bas.bg

Stela Dimitrova Zhelezova
Institute of Mathematics
Bulgarian Academy of Sciences
P.O. Box 323
5000 Veliko Tarnovo, Bulgaria
e-mail: stela@moi.math.bas.bg

КЛАСИФИКАЦИЯ НА 2-СПРЕДОВЕТЕ НА $PG(5,2)$

Светлана Тодорова Топалова, Стела Димитрова Железова

С помощта на компютър построяваме 2-спредовете на $PG(5,2)$ и ги тестваме за еквивалентност по метод, използващ специфичните свойства на групата от автоморфизми на проективната геометрия. Получаваме, че с точност до еквивалентност съществува един единствен 2-спред на $PG(5,2)$. Групата от автоморфизми, която го запазва, е от ред 10584.