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PCIMs in constructing doubly resolvable designs

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Abstract. Resolvable designs with parallel classes of size q correspond to equidistant codes over Z(q), while doubly resolvable 2-(v,k,λ) designs also correspond to Kirkman squares $KS_k(v;1,\lambda)$. In this work we construct doubly resolvable BIBDs using the intersection possibilities between the parallel classes. We investigate the structure of the resolutions of a design and make conclusions about the structure of the resolutions of doubly resolvable designs. Next we consider by computer search only structures, which can produce doubly resolvable designs. In this way we classify doubly resolvable 2-(16,8,7), 2-(24,12,11), 2-(28,14,13), 2-(16,8,14), 2-(32,16,15), 2-(18,9,16) designs and resolvable 2-(12,6,10) designs.

1 Introduction

For the basic concepts and notations concerning combinatorial designs and their resolvability refer, for instance, to [1], [2], [3], [5], [18].

Let $V = \{P_i\}_{i=1}^{v}$ be a finite set of *points*, and $\mathcal{B} = \{B_j\}_{j=1}^{b}$ – a finite collection of k-element subsets of V, called *blocks*. We say that $D = (V, \mathcal{B})$ is a *design* with parameters t-(v,k,λ), if any t-subset of V is contained in exactly λ blocks of $\mathcal{B}$.

Two designs are *isomorphic* if there exists a one-to-one correspondence between the point and block sets of the first design and respectively, the point and block sets of the second design, and if this one-to-one correspondence does not change the incidence.

An *automorphism* of the design is a permutation of the points that transforms the blocks into blocks.

One of the most important properties of a design is its resolvability. The design is *resolvable* if it has at least one resolution.

A *resolution* is a partition of the blocks into subsets called *parallel classes* such that each point is in exactly one block of each parallel class. A parallel class contains v/k blocks and a resolution $\mathcal{R}$ consists of $r = (b * k/v)$ parallel classes, $\mathcal{R} = \mathcal{R}_i, ..., \mathcal{R}_r$.

Two resolutions are isomorphic if there exists an automorphism of the design transforming each parallel class of the first resolution into a parallel class of the second one.

A parallel class T is *orthogonal* to the resolution $\mathcal{R}$ if $T \cap \mathcal{R}_i$ contains 0 or 1 block for each $1 \leq i \leq r$. Let $\mathcal{R} = \mathcal{R}_i, ..., \mathcal{R}_r$ and $T = T_i, ..., T_r$ be resolutions of the same design. These two resolutions are *orthogonal* if the number of blocks in $\mathcal{R}_i \cap T_j$ is either 0 or 1 for all $1 \leq i, j \leq r$. When a design has at least two orthogonal resolutions, it is *doubly resolvable.*

A Kirkman square with index λ, latinicity μ, block size k, and v points, $KS_k(v;\mu,\lambda)$ is a $t \times t$ array ($t = \lambda(v-1)/\mu(k-1)$) defined on a set V such that: every point of V is contained in precisely μ cells of each row and column; each cell of the array is either empty or contains a k-subset of V; the collection of blocks obtained from the non-empty cells of the array is a (v,k,λ) BIBD. For μ=1, the existence of a $KS_k(v;\mu,\lambda)$ is equivalent to the existence of a doubly resolvable 2-(v,k,λ) design.

The existence question for $KS_k(v;\mu,\lambda)$ has been completely settled for $k = 2$ and $\mu = 1$ [8]. The existence of $KS_3(v;1,2)$ for all $v \equiv 3(mod)12$ is proved in [9]. There are some particular results for $k \geq 3$, $\mu = 1$ in [7], [4], [14], [15].

The intersection of parallel classes is used in [11], [12] and [6] for the construction of resolvable designs. In [11] and [12] it is used to produce initial structures of the resolution of the designs and in [6] for partial verification of the classification of resolvable 2-(14,7,12) designs.

Double resolvability sets additional restrictions on the intersection between parallel classes. In the present work we use this to remove some of the constructions which cannot be doubly resolvable. It is especially effective for design resolutions, which have 2 blocks in the parallel class. Then by computer search we construct point by point resolutions which can have orthogonal partners and check partial solutions of more than 2/3 of the points for double resolvability [16]. This check consists of an attempt to construct a corresponding Kirkman square [17].

2 Structure of a resolution of a design in terms of PCIM

For two parallel classes $c = \{C_1, ..., C_n\}$ and $t = \{T_1, ..., T_n\}$ on v points (where $n = v/k$ is the number of the blocks in the parallel class) we define the intersection between them as an $n \times n$ matrix. This matrix is called the parallel classes intersection matrix(PCIM)[6], [11], [12]. In this matrix $P(c,t)_{n\times n} = (p_{ij}(c,t))$, where $p_{ij}(c,t)$ is the intersection size of blocks $C_i \in c$ and $T_j \in t$. For any two parallel classes c and t, $p_{i1}(c,t) + p_{i2}(c,t) + \ldots + p_{in}(c,t) = k$ for $i = 1, 2, \ldots, n$. In this way if we fix the first parallel class of the resolution as

$$\begin{pmatrix} k & 0 & \dots & 0 \\ 0 & k & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & k \end{pmatrix}$$

by computer search we produce all possible inequivalent PCIMs for the second parallel class. Let m be their number. We denote by x_i the number of classes of the resolution, for which the intersection matrix between the first class and them is equivalent to the i-th PCIM, $i = 1, 2, ..., m$.

Since a resolvable $2-(v, k, \lambda)$ design has r parallel classes, the first parallel class meets other classes in (r-1) PCIMs.

$$\sum_{i=1}^{m} x_i = r - 1 \tag{1}$$

Denote by ψ_i the number of pairs of points, which are contained in a block of the first parallel class and in a block of a parallel class corresponding to the i-th PCIM. Then

$$\sum_{i=1}^{m} \psi_i . x_i = v(k-1)(\lambda-1)/2 \tag{2}$$

Solutions of these equations give the structure of the resolutions of a design.

3 Double resolvability restrictions

The existence of a Kirkman square requires at least two resolutions of the design, such that the parallel classes of both of them have at most one common block. We can analyse the obtained solutions and make conclusions about possible double resolvability.

According to the type of the PCIMs we can say if the blocks of the parallel classes of an initial resolution can be partitioned in different parallel classes of a partner resolution. The condition for this is the occurrence of suitable PCIMs, allowing a combination of blocks of different parallel classes. We consider all types of PCIMs and find out solutions of the equations (1) and (2) which can give an orthogonal resolution to the initial one.

In the case of resolutions with two blocks in a parallel class, we look for PCIMs corresponding to classes with at least one block which intersects one of the blocks of the first parallel class in k points. If no such PCIM exists, doubly resolvable designs cannot be obtained.

4 Computer search and results

According to the obtained PCIM patterns, we construct the resolutions of a design point by point. (Actually we construct word by word the corresponding equidistant code [13].) We apply equivalence test after each point, and double resolvability test if the partial solution is of more than 2/3 of the points [16]. This should be done because the solutions of equations (1) and (2) and the double resolvability restrictions give only a necessary condition for the possible existence of orthogonal resolutions. In this test we try to find a corresponding Kirkman square with $\mu = 1$. At first we find all possible variants for a parallel class of an orthogonal resolution by choosing disjoint blocks from different parallel classes of the initial resolution until all points are covered. Then we try to combine different variants for new parallel classes in a Kirkman square [17].

For the 2-(16,8,7) designs $k = 8$, so the PCIMs are:

$$\begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix} \begin{pmatrix} 7 & 1 \\ 1 & 7 \end{pmatrix} \begin{pmatrix} 6 & 2 \\ 2 & 6 \end{pmatrix} \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix}$$

In this case equations (1) and (2) give:

$$x_1 + x_2 + x_3 + x_4 + x_5 = 14$$
$$56.x_1 + 42.x_2 + 32.x_3 + 26.x_4 + 24.x_5 = 336$$

There is only one solution - $x_1 = x_2 = x_3 = x_4 = 0$, $x_5 = 14$. It shows us that the design cannot be doubly resolvable - there is no parallel class, which intersects a block of the first one in k points. Since the blocks of the first parallel class have to be in different parallel classes of the orthogonal resolution, they cannot be combined with blocks from other parallel classes of the initial resolution.

We use this solution for the structure of the resolutions of this design to restrict the search space and we find 5 nonisomorphic resolutions. We verify this result constructing by exhaustive search all the resolutions of the design with these parameters.

For the 2-(16,8,14) designs PCIMs are the same as for the 2-(16,8,7), so we have:

$$x_1 + x_2 + x_3 + x_4 + x_5 = 29$$
$$56.x_1 + 42.x_2 + 32.x_3 + 26.x_4 + 24.x_5 = 728$$

with the following 8 solutions:

x_1	1	0	0	0	0	0	0	0
x_2	0	1	0	1	0	0	0	0
x_3	0	0	0	1	1	3	2	4
x_4	0	7	16	3	12	4	8	0
x_5	28	21	13	24	16	22	19	25

The first one only can give doubly resolvable designs. We investigate it by computer search and we find 5 resolutions of doubly resolvable designs and 1895 resolutions of resolvable designs.

For the 2-(12,6,10)designs $k = 6$ and the PCIMs are:

$$\begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix} \begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}$$

In this case it holds:

$$x_1 + x_2 + x_3 + x_4 = 21$$
$$30.x_1 + 20.x_2 + 14.x_3 + 12.x_4 = 270$$

with 3 solutions

x_1	1	0	0
x_2	0	2	1
x_3	0	1	5
x_4	20	18	15

We investigate the first one as possibly doubly resolvable and we find 1 resolution of 1 doubly resolvable design. This result coincides with our previous results in [14]. We also find altogether 545 nonisomorphic resolutions of resolvable designs for all three solutions (the previous bound was ≥ 400 [10]).

There are two blocks in the parallel class of 2-(24,12,11), 2-(28,14,13), 2-(32,16,15), and 2-(18,9,16) designs. So the PCIMs are:

$$\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} \begin{pmatrix} k-1 & 1 \\ 1 & k-1 \end{pmatrix} \begin{pmatrix} k-2 & 2 \\ 2 & k-2 \end{pmatrix} \begin{pmatrix} k-i & i \\ i & k-i \end{pmatrix} \cdots \begin{pmatrix} \lfloor k/2 \rfloor & \lfloor k/2 \rfloor \\ \lfloor k/2 \rfloor & \lfloor k/2 \rfloor \end{pmatrix}$$

As a result of equations (1) and (2) there are only solutions without a PCIM corresponding to a class with at least one block, which intersects one of the blocks of the first parallel class in k points. That is why such solutions cannot give a doubly resolvable design. So we can conclude that doubly resolvable design with parameters 2-(24,12,11), 2-(28,14,13), 2-(32,16,15), and 2-(18,9,16) do not exist.

The obtained results are summarized in Table 1. The first column shows the number of the design in the table of [10], the second its parameters. In the third column we present the number of nonisomorphic resolutions "Nr", followed by $*$ if the result is not ours, but from [10]. In the column "drNr" we present the number of nonisomorphic resolutions of the doubly resolvable designs with these parameters.

No	BIBD	Nr	drNr
130	(16,8,7)	5	0
319	(12,6,10)	545	1
346	(24,12,11)	$\geq 129^*$	0
499	(28,14,13)	$\geq 4^*$	0
618	(16,8,14)	≥ 1895	5
668	(32,16,15)	$\geq 1^*$	0
791	(18,9,16)	$\geq 6^*$	0

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