TENTH INTERNATIONAL WORKSHOP ON

ALGEBRAIC AND COMBINATORIAL CODING THEORY

Proceedings

3-9 September 2006, Zvenigorod, Russia

On the structure of doubly-resolvable 2-(21,3,1) designs

Svetlana Topalova *, Stela Zhelezova
Institute of Mathematics and Informatics, BAS, Bulgaria,
svetlana@moi.math.bas.bg, st_zhelezova@abv.bg

Abstract

Resolvable designs with parallel classes of size q correspond to equidistant codes over Z(q), while doubly resolvable 2-(v,k,λ) designs also correspond to Kirkman squares $KS_k(v;1,\lambda)$. The problem of the existence of a doubly resolvable Steiner triple system of order 21 (STS(21) or 2-(21,3,1) design) is still open with 21 being the smallest value for v, for which it is not known if a doubly resolvable $STS(v)$ exists or not. In this work we make some notes on the structure of a doubly resolvable STS(21).

1 Introduction

For the basic concepts and notations concerning combinatorial designs refer, for instance, to [1], [4], [21].

Let $V = \{P_i\}_{i=1}^{v}$ be a finite set of *points*, and $\mathcal{B} = \{B_j\}_{j=1}^{b}$ – a finite collection of k-element subsets of V, called *blocks*. We say that $D = (V, \mathcal{B})$ is a *design* with parameters t-(v,k,λ), if any t-subset of V is contained in exactly λ blocks of $\mathcal{B}$.

Two designs are *isomorphic* if there exists a one-to-one correspondence between the point and block sets of the first design and respectively, the point and block sets of the second design, and if this one-to-one correspondence does not change the incidence.

An *automorphism* of the design is a permutation of the points that transforms the blocks into blocks.

One of the most important properties of a design is its resolvability. The design is *resolvable* if it has at least one resolution.

A *resolution* is a partition of the blocks into subsets called *parallel classes* such that each point is in exactly one block of each parallel class. Two resolutions are isomorphic if there exists an automorphism of the design transforming each parallel class of the first resolution into a parallel class of the second one.

*This work was partially supported by the Bulgarian National Science Fund under Contract No I-1301/2003.

There are already quite a lot of works on the existence or classification of resolvable BIBDs with definite parameters, see for instance [2], [8], [10], [17], [18], [19]. It is interesting to point out that in some recent works the classification was only possible after using parameter-specific restrictions on the corresponding equidistant codes.

Double resolvability is of particular interest. A 2-(v,k,λ) design is *doubly-resolvable* if it has two distinct resolutions (partner resolutions) such that each pair of parallel classes, one of the first, and the other of the second resolution, have at most one common block. Papers on doubly-resolvable designs mainly deal with the setting of the existence problem [5], [6], [7], [16].

A Steiner triple system of order v ($STS(v)$) is a $2-(v,3,1)$ design. STS(v)s exist for $v \equiv 1$ or $3(mod6)$. A resolvable $STS(v)$ is called Kirkman triple system of order v ($KTS(v)$) and can exist for $v \equiv 3(mod6)$. A doubly resolvable $STS(v)$ does not exist when $v \in \{9,15\}$, but exists for all $v > 21$ with $v \equiv 3(mod6)$ with 23 possible exceptions [5]. The smallest possible exception occurs when $v = 21$, so that the smallest known (nontrivial) doubly resolvable $STS(v)$ has $v = 27$.

Steiner triple systems are fully classified for $v <= 19$. For the next value $v = 21$ a complete classification is currently out of reach [13], but various classification results on STS(21) with additional properties exist [9], [11], [12], [14], [20], [15], [22]. All these authors also test the obtained STS(21)s for resolvability. However, the number of the known KTS(21) was relatively small until 63745 KTS(21) were constructed in [3]. These include all KTS(21) possessing nontrivial automorphisms. The authors of [3] have also made some investigations on the structure of these KTS(21)s. They solved the problem posed in [1] for the determination of a quadrilateral-free KTS(21), i.e. they found four quadrilateral-free KTS(21)s (a quadrilateral or Pasch configuration is a set of four triples on six elements which pairwise intersect in one element each). Yet none of these 63745 KTS(21)s is doubly resolvable.

We approach the problem by trying to construct only designs which are doubly resolvable. In this work we consider some peculiarities of the structure of a KTS(21), which help a lot for making the search space smaller. We split the problem into several cases, and for one subcase of them we have checked by computer that it doesn't lead to a doubly resolvable STS(21).

2 On the structure of a KTS(21)

Lemma 1 *Let $D = (V,B)$ be a $KTS(21)$. Consider three of the triples of one parallel class. They define a 9-element subset V_9 of V. Let t_9 be the number of triples on the points of V_9. Let $V_{12} = V \setminus V_9$.Let t_{12} be the number of triples on the points of V_{12}. Then*

a. $3 \le t_9 \le 12$
b. $t_{12} = 16 - t_9$
c. $4 \le t_{12} \le 13$

Proof. a.There are at least three triples of the chosen parallel class, while 12 triples form an STS(9).

b. Denote by a_s the number of s-subsets on the points of V_9, where $s = 0, 1, 2, 3$. The design has 70 blocks and thus $a_0 + a_1 + a_2 + a_3 = 70$. The number of pairs of points, such that one of them is from V_9, and the other one from V_{12}, should be 9.12, and thus $2a_1 + 2a_2 = 108$. It follows that $a_0 + a_3 = 16$. Yet a_3 is another name for t_9 and a_0 for t_{12}.

c. Follows from a. and b.

Theorem 1 *Let $D = (V, B)$ be a $KTS(21)$. Then*

1. There exists a 9-element subset V_9 of V, such that there are 3 or 4 triples on it, and three of them are nonintersecting.

2. V_9 is contained in a 12-element subset V_{12} of V, such that there are 11, 12, or 13 triples on the points of V_{12}, at least 4 of them being nonintersecting.

Proof.

1. Consider the triples $t_a, a = 1, 2, ..., 7$ of one parallel class and define a 12-element subset $V'_{12} = t_1 \bigcup t_2 \bigcup t_3 \bigcup t_4$. Consider 2 cases:

a) There are 12 or 13 triples on V'_{12}. Then by Lemma 1b there are 4 or 3 triples on $V_9 = V \setminus V'_{12}$.

b) There are at most 11 triples on V'_{12}, i.e. the 4 triples t_1, t_2, t_3, t_4 and at most 7 other triples. Let t_a have a common point with n_a of these 7 triples (a=1,2,3,4). Then $n_1 + n_2 + n_3 + n_4 = 7.3 = 21$. Without loss of generality we can assume that n_4 is the smallest of these 4 numbers. Then the maximal value of n_4 is 5 (i.e. 5+5+5+6=21). Each one of these n_4 triples is in one of the 3 sets $t_1 \bigcup t_2 \bigcup t_4$, $t_1 \bigcup t_3 \bigcup t_4$, or $t_2 \bigcup t_3 \bigcup t_4$. Denote by V_9 this one of the three sets, which contains the least number of triples. Then it contains at most one of them (i.e. 2+2+1=5), i.e. the whole number of triples on V_9 is 3 or 4.

2. Let $V_9 = t_1 \bigcup t_2 \bigcup t_3$. Without loss of generality we can assume that there are at most 4 triples on V_9. Consider 2 cases:

a) There are 3 triples on V_9. Then there are 27 more pairs of points on V_9, which form a triple together with a point outside V_9. Let m_a be the number of triples with a point of $t_a, a = 4, 5, 6, 7$. Then $m_4 + m_5 + m_6 + m_7 = 27$. Assume m_4 is the maximal among these 4 numbers. Then $m_4 = 7, 8$, or 9 (i.e. 7+7+7+6=27). Define $V_{12} = V_9 \bigcup m_4$. Then V_{12} contains 11,12, or 13 triples.

b) There are 4 triples on V_9. Then there are 24 more pairs of points on V_9, which form a triple together with a point outside V_9. Let m_a be the number of triples with a point of $t_a, a = 4, 5, 6, 7$. $m_4 + m_5 + m_6 + m_7 = 24$. Assume m_4 is the maximal among these 4 numbers. Then $m_4 = 6, 7, 8$. Define $V_{12} = V_9 \bigcup m_4$. Then V_{12} contains 11,12, or 13 triples.

3 On the structure of a doubly resolvable STS(21)

Theorem 2 *Let $D = (V, B)$ be a doubly resolvable $STS(21)$, and let R and R' be two partner resolutions. Consider three of the triples of one parallel class C_1 of R. They define a 9-element subset V_9 of V. Then there are at most 11 triples on the points of V_9.*

Proof. Suppose there are 12 triples on V_9 as Lemma 1a allows. The parallel classes are 10, so there is at least one parallel class of R' with at least two triples on V_9, and thus with also at least two blocks without points on V_9. But all the 4 blocks without points on V_9 are in C_1, and thus no two of them can be in one of the same class of R'. $\diamond$

This result coincides with [15], whose authors claim that no STS(21) having a Steiner triple subsystem is doubly resolvable.

4 Computer tests

Let us consider only the set V_n of n points of a design resolution and all the design blocks redefined only by the incidence with V_n. We experimented how the double-resolvability test works on this structure. If the whole design is doubly-resolvable, any such structure is obviously doubly-resolvable too. If the design is not doubly-resolvable, and we take less than two thirds of its points, we usually obtain a doubly-resolvable structure.

We first developed an algorithm for random generation of the incidence of the blocks of different KTS(21)s with the set V_{15} defined on five triples of one of its parallel classes. None of them were doubly resolvable. So the problem is how to construct all 14-row parts of the incidence matrix of a KTS(21), which if extended will give all doubly resolvable STS(21)s, if such exist. Even with the strong restrictions from the above two sections, this still remains a very hard computational problem.

Let $D = (V, B)$ be a $KTS(21)$. Consider the triples $t_a, a = 1, 2, ..., 7$ of one parallel class. By Theorem 1 there are 5 different cases for the incidence of the blocks with $V_{12} = t_1 \bigcup t_2 \bigcup t_3 \bigcup t_4$. In three of them there are 3 triples incident with $V_9 = t_1 \bigcup t_2 \bigcup t_3$ and respectively 11,12, or 13 triples incident with V_{12}. In the other two there are 4 triples incident with V_9 and respectively 11 or 12 triples incident with V_{12} (it can be proved that 4 triples on V_9 and 13 triples on V_{12} is equivalent to 3 triples on V_9 and 12 triples on V_{12}).

We have checked that no doubly resolvable STS(21) can be obtained from one subcase of the 3 triples on V_9 and 13 triples on V_{12} case. In addition for the KST(21)s we require that

- there are 9 parallel classes of one and the same type up to the 6-th, 9-th, 12-th and 15-th row. A class has one block incident with a pair of V_6, 4 blocks incident with one point of V_6, and 2 blocks incident with no point of V_6; three blocks incident with a pair of V_9, 3 blocks incident with one point of V_9, and 1 block incident with no point of V_9; three blocks incident with a pair of V_{12}, 3 blocks incident with one point of V_{12}, and 1 block incident with a triple of V_{12}, two blocks incident with a triple of V_{15}, 4 blocks incident with two points of V_{15}, and 1 block incident with one point of V_{15}.

- there are 6 triples on V_{10}, 9 on V_{11}, 13 on V_{12}, 16 on V_{13},and 19 triples incident with V_{14}.

- any two triples on $t_i \bigcup t_j \bigcup t_k$ $(i, j, k = 1, 2, ..., 5)$ have no common point.
- the blocks of a parallel class contain at most one pair of points from $t_i \bigcup t_j$ $(i, j = 1, 2, ..., 5)$.
- We require that none of the triples on V_{15} contains a pair of points from $t_4 \bigcup t_5$.

We construct the design resolutions in lexicographic order point by point (To do it faster we actually construct word by word the corresponding equidistant code). After each point we apply a test for equivalence of the partial solution to a previously generated one, and a double-resolvability test after the 12-th point.

References

[1] Beth Th., Jungnickel D., Lenz H., *Design Theory*, Cambridge University Press, 1993.

[2] Bose R., A note on the resolvability of balanced incomplete block designs, *Sankhya* 6 (1942), 105-110.

[3] Cohen M., Colbourn Ch., Ives Lee A. , Ling A., Kirkman triple systems of order 21 with nontrivial automorphism group. Math. Comput. 71(238): 873-881 (2002)

[4] Colbourn C.J. and Dinitz J.H. (Eds.), *The CRC Handbook of Combinatorial Designs*, CRC Press, Boca Raton, FL., CRC Press, 1996.

[5] Colbourn Ch., Lamken E., Ling A., Mills W., The existence of Kirkman squares - doubly resolvable (v,3,1)-BIBDs, *Designs, Codes and Cryptography* 26 (2002), 169-196.

[6] Curran D., Vanstone S., Doubly resolvable designs from generalized Bashkar Rao designs, *Discrete Mathematics* 73 (1988/89) 49-63.

[7] Jimbo M., Vanstone S., Recursive constructions for resolvable and doubly resolvable 1-rotational Steiner 2-designs, *Utilitas Mathematica*, vol. 26 (2002), 155-168.

[8] Kageyama S., A survey of resolvable solutions of balanced incomplete block designs, Int. Stat. Rev vol. 40, No 3 (1972), 269-273.

[9] S.N. Kapralov and S. Topalova, On the Steiner triple systems of order 21 with automorphisms of order 3, in: Proc. Third International Workshop on Algebraic and Combinatorial Coding Theory (Voneshta Voda, Bulgaria, 22–28 June, 1992) 105–108.

[10] Kaski P., Morales L., Östergård P., Rosenblueth D., Velarde C., Classification of resolvable 2-(14,7,12) and 3-(14,7,5) designs, *Journal of Combinatorial Mathematics and Combinatorial Computing* 47 (2003), 65-74.

[11] R.A. Mathon, K.T. Phelps, and A. Rosa, A class of Steiner triple systems of order 21 and associated Kirkman systems, Math. Comp. 37 (1981) 209–222; and 64 (1995) 1355–1356.

[12] R. Mathon and A. Rosa, The 4-rotational Steiner and Kirkman triple systems of order 21, Ars Combin. 17A (1984) 241–250.

[13] Mathon R., Rosa A., 2-(v,k,λ) designs of small order, *The CRC Handbook of Combinatorial Designs*, Boca Raton, FL., CRC Press, 1996, 3-41.

[14] Kaski P., Östergård P., Enumeration of 2-(9,3,λ) designs and their resolutions, *Designs, Codes and Cryptography* 27 (2002), 131-137.

[15] Kaski P., Östergård P., Topalova S., Zlatarski R., Steiner Triple Systems of Order 19 and 21 with Subsystems of Order 7, *Discrete Mathematics*, to appear

[16] Lamken E., Vanstone S., The existence of a class of Kirkman squares of index 2, *J. Ausral. Math. Soc. (Series A)* 44 (1988), 33-41.

[17] Morales L., Velarde C., A complete classification of (12,4,3)-RBIBDs, Journal of Combinatorial Designs vol.9,issue 6 (2001), 385-400.

[18] Morales L., Velarde C., Enumeration of resolvable 2-(10,5,16) and 3-(10,5,6) designs, *Journal of Combinatorial Designs* vol.13, issue 2, 108-119.

[19] Takeuchi K.,A table of difference sets generating balanced incomplete block designs, *Rev. Inst. Intern. Stat.* 30:3 (1962), 361-366.

[20] V.D. Tonchev, Steiner triple systems of order 21 with automorphisms of order 7, Ars Combin. 23 (1987) 93–96; and 39 (1995) 3.

[21] Tonchev V.D., *Combinatorial configurations*, Longman Scientific and Technical, New York, 1988.

[22] Topalova S., STS(21) with Automorphisms of order 3 with 3 fixed points and 7 fixed blocks, Mathematica Balkanica, vol.18 (2004), Num 1-2, 215-220.