Parallelisms of $PG(3, 5)$ with automorphisms of order 13 \(^1\)

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Dedicated to the memory of Professor Stefan Dodunekov

Abstract. A spread is a set of lines of $PG(n, q)$, which partition the point set. A parallelism is a partition of the set of lines by spreads. Some constructions of constant dimension codes that contain lifted MRD codes are based on parallelisms of projective spaces. We construct all (321) parallelisms in $PG(3, 5)$ with automorphisms of order 13 and find the order of their full automorphism groups.

1 Introduction

The relation to translation planes \([4]\) has been one of the main reasons for the consideration of $t$-spreads and $t$-parallelisms. There are applications of spreads and parallelisms in Coding Theory too. For instance, parallelisms are used in constructions of constant dimension error-correcting codes that contain lifted MRD codes \([5]\). The relation of parallelisms to resolutions of Steiner systems leads to a cryptographic usage for anonymous $(2, q + 1)$-threshold schemes \([13]\).

For the basic concepts and notations concerning spreads and parallelisms of projective spaces, refer, for instance, to \([4]\), \([7]\) or \([15]\).

A $t$-spread in $PG(n, q)$ is a set of distinct $t$-dimensional subspaces which partition the point set. A $t$-parallelism is a partition of the set of $t$-dimensional subspaces by $t$-spreads. Usually 1-spreads (1-parallelisms) are called line spreads (line parallelisms) or just spreads (parallelisms). There can be line spreads and parallelisms if $n$ is odd.

Two parallelisms are isomorphic if there exists an automorphism of the projective space which maps each spread of the first parallelism to a spread of the second one.

A subgroup of the automorphism group of the projective space which maps each spread of the parallelism to a spread of the same parallelism is called automorphism group of the parallelism. A parallelism is called transitive if it has an automorphism group which is transitive on the spreads.

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Let $V = \{P_i\}_{i=1}^v$ be a finite set of points, and $B = \{B_j\}_{j=1}^b$ a finite collection of $k$-element subsets of $V$, called blocks. $D = (V,B)$ is a 2-design with parameters 2-$(v,k,\lambda)$ if any 2-subset of $V$ is contained in exactly $\lambda$ blocks of $B$. A parallel class is a partition of the point set of the design by blocks. A resolution of the design is a partition of the collection of blocks by parallel classes.

The incidence of the points and $t$-dimensional subspaces of $\text{PG}(n, q)$ defines a 2-design (see for instance [15, 2.35-2.36]). There is a one-to-one correspondence between the parallelisms of $\text{PG}(3, 5)$ and the resolutions of the 2-(156,6,1) design of its points and lines.

A construction of parallelisms in $\text{PG}(n, 2)$ is presented by Zaicev, Zinoviev and Semakov [19] and independently by Baker [1], and in $\text{PG}(2^n - 1, q)$ by Beutelspacher [2]. Several constructions are known in $\text{PG}(3, q)$ due to Dennis-ton [3], Johnson [7], Penttila and Williams [9].

Several computer aided classifications of $t$-parallelisms are available too. Prince classified parallelisms of $\text{PG}(3, 3)$ with automorphisms of order 5 [10], and parallelisms of $\text{PG}(3, 5)$ with automorphisms of order 31 [11]. Stinson and Vanstone classified parallelisms of $\text{PG}(5, 2)$ with a full automorphism group of order 155 [14], Sarmiento with a point-transitive cyclic group of order 63 [12], and Zhelezova with a cyclic group of order 31 [20]. Topalova and Zhelezova classified parallelisms of $\text{PG}(3, 4)$ with automorphisms of orders 7 [17] and 5 [18], and 2-parallelisms of $\text{PG}(5, 2)$ with automorphisms of order 31 [16].

A regulus of $\text{PG}(3,q)$ is a set $R$ of $q + 1$ mutually skew lines such that any line intersecting three elements of $R$ intersects all elements of $R$. A spread $S$ of $\text{PG}(3, q)$ is regular if for every three distinct elements of $S$, the unique regulus determined by them is a subset of $S$. A parallelism is regular if all its spreads are regular. Each regular parallelism of $\text{PG}(3, q)$ corresponds to a translation plane of order $q^4$ [8].

In 1998 Prince constructed all 45 transitive parallelisms of $\text{PG}(3, 5)$. They are cyclic and possess automorphisms of order 31. Among them there are two regular parallelisms. A little later in 1998 Penttila and Williams constructed two regular cyclic parallelisms of $\text{PG}(3, q)$ for any $q \equiv 2 \pmod{3}$ [9]. All presently known examples of regular parallelisms are among them and the existence of other regular parallelisms is an open question.

We check all the constructed parallelisms for regularity. There are no regular ones among them.

Our C++ programmes performing the computer computations are based on the exhaustive backtrack search techniques. To filter away isomorphic parallelisms, we find the normalizer of the chosen subgroup of order 13 in the automorphism group of the projective space.

We use design approach to the problem. We actually make all the computations on the related to $\text{PG}(3, 5)$ designs, namely, we choose the 26 spread elements among the 806 blocks of the 2-(156,6,1) point-line design, and construct the parallelisms as its resolutions. We find a generating set of the automorphism group of $\text{PG}(3, 5)$ as generating set of the automorphism group of
the related 2-(156, 31, 6) point-hyperplane design.

2 Construction of the parallelisms

There are 156 points and 806 lines in $PG(3, 5)$. Denote by $G$ the full automorphism group of $PG(3, 5)$. It is of order $29016000000 = 2^9 \cdot 3^2 \cdot 5^6 \cdot 13 \cdot 31$. A spread has 26 lines which partition the point set and a parallelism has 31 spreads.

To construct $PG(3, 5)$ we use the 4-dimensional vector space over $GF(5)$. The points of $PG(3, 5)$ are then all 4-dimensional vectors $(v_1, v_2, v_3, v_4)$ over $GF(5)$ such that $v_i = 1$ if $i$ is the maximum index for which $v_i \neq 0$. We sort these 156 vectors in ascending lexicographic order and then assign them numbers such that $(1, 0, 0, 0)$ is number 1, and $(4, 4, 4, 1)$ number 156. We then construct the related designs and find the generators of their full automorphism group $G$.

By Sylow’s Theorems all subgroups of order 13 are conjugate, and we can choose an arbitrary one of them. We use GAP [6] to find a Sylow subgroup of order 13 and denote it $G_{13}$.

We sort the 806 lines (blocks of the 2-(156, 6, 1) design) in lexicographic order defined on the numbers of the points they contain and assign to each line a number according to this order. $G_{13}$ partitions the points in 12 orbits of length 13 and the lines in 62 orbits of length 13. We look for line orbits whose lines contain each point at most once. This holds for 26 line orbits, and therefore $G_{13}$ cannot fix more than 13 spreads. It follows that parallelisms with an automorphism of order 13 have

- 5 fixed spreads made of 2 line orbits and
- 2 orbits of 13 spreads each.

We construct the spreads by backtrack search. The first element in each spread is a line, containing point 1. If there are already $m$ elements in the spread, we choose the $m + 1$-st one among the lines containing the first point which is in none of the $m$ spread elements and we take in consideration the orbits of the spread lines. Any spread we construct is lexicographically greater than the ones constructed before it. Each spread determines all the other spreads of its orbit. We call the first spread orbit leader. To obtain a parallelism with $G_{13}$ we need to construct only the seven orbit leaders.

The rejection of isomorphic solutions is an important part of the computation. We construct only parallelisms which are invariant under $G_{13}$. Therefore we have to check if there is some permutation $\varphi \in G$ such that it maps a parallelism with $G_{13}$ to another parallelism with $G_{13}$.

Let $\mathcal{P}$ and $\mathcal{P}'$ be two of the constructed parallelisms such that $\mathcal{P}' = \varphi \mathcal{P}$. Let $\alpha \in G_{13}$. Then $\varphi \mathcal{P} = \alpha \varphi \mathcal{P}$ and thus $\mathcal{P} = \varphi^{-1} \alpha \varphi \mathcal{P}$, namely $\varphi^{-1} \alpha \varphi$ is also an automorphism of $\mathcal{P}$. That is why $\mathcal{P}$ is invariant both under $G_{13}$ and
under $\varphi^{-1}G_{13}\varphi = G_{13}$, then $\varphi$ is in the normalizer $N(G_{13})$ of $G_{13}$ in $G$, which is defined as $N(G_{13}) = \{g \in G \mid gG_{13}g^{-1} = G_{13}\}$. If $\varphi$ is not in the normalizer, then $\varphi^{-1}G_{13}\varphi$ is a conjugate subgroup and since $\varphi$ is not an automorphism of $\mathcal{P}$, there must exist an automorphism $\psi$ of $\mathcal{P}$ such that $\varphi^{-1}G_{13}\varphi = \psi G_{13}\psi^{-1}$. Then $G_{13} = \varphi \psi G_{13}\psi^{-1}\varphi^{-1}$ and therefore $\varphi \psi \in N(G_{13})$. Since $\varphi \psi \mathcal{P} = \varphi \mathcal{P}$, to establish isomorphism of two of the constructed parallelisms it is enough to consider only automorphisms from $N(G_{13})$.

For each parallelism $\mathcal{P}$ we obtain, we check if an automorphism of $N(G_{13})$ maps it to a parallelism with a lexicographically smaller orbit leader sequence, and drop it if so.

If some element of $N(G_{13})$ maps $\mathcal{P}$ to itself, it is its automorphism. So by the isomorphism test we also obtain some of the automorphisms of this parallelism. If $\mathcal{P}$ has an automorphism $\psi \notin N(G_{13})$, then it is invariant under $\psi G_{13}\psi^{-1}$. That is why we check if $\mathcal{P}$ is invariant under some conjugate subgroup of $G_{13}$. In fact we establish that none of the constructed parallelisms are invariant under conjugate subgroups.

3 Classification results

We obtain 321 non-isomorphic parallelisms. All of them have automorphisms of order 13 only. We find out that there are no regular ones among them.

Since software mistakes are always possible, we obtained the number of non-isomorphic parallelisms by two different C++ programmes (written by the two authors).

References


