Corrigendum

In the following conference paper we erroneously announced that there are no transitive deficiency one parallelisms in $PG(3,4)$. This was the result of an error in our computation of the orders of the automorphism groups. In fact there are four transitive deficiency one parallelisms in $PG(3,4)$. More details are given in the paper:

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Parallelisms of $PG(3,4)$ with automorphisms of order 5

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Abstract. A spread is a set of lines of $PG(d,q)$, which partition the point set. A parallelism is a partition of the set of lines by spreads. Some constructions of constant dimension codes that contain lifted MRD codes are based on parallelisms of projective spaces.
A parallelism is transitive if it has an automorphism group which is transitive on the spreads. A parallelism is point-transitive if it has an automorphism group which is transitive on the points. If the automorphism group fixes one spread and is transitive on the remaining spreads, the parallelism corresponds to a transitive deficiency one partial parallelism.
In $PG(3,4)$ there are no transitive parallelisms. No examples of point-transitive and no examples of transitive deficiency one parallelisms of $PG(3,4)$ are known. We construct all 28270 nonisomorphic parallelisms with automorphisms of order 5. None of them is point-transitive. There are 28100 ones with an automorphism group fixing exactly one spread, but none of them is transitive on the remaining spreads. We conclude that there are no point-transitive parallelisms and no transitive deficiency one parallelisms in $PG(3,4)$.

1 Introduction

For the basic concepts and notations concerning combinatorial designs, projective spaces, spreads and parallelisms, refer, for instance, to [1], [2], [5], [8], [11], or [19], and for the application of parallelisms in constructions of constant dimension codes that contain lifted MRD codes refer to [16] and [17].

A *spread* in $PG(d,q)$ is a set of lines which partition the point set. A *parallelism* is a partition of the set of lines by spreads. There can be line spreads and parallelisms if d is odd. A deficiency one parallelism is a partial parallelism with one spread less than the parallelism. Each deficiency one parallelism can be uniquely extended to a parallelism.

A subgroup of the automorphism group of the projective space, which maps each spread of the parallelism to a spread of the same parallelism is called *automorphism group* of the parallelism. A (partial) parallelism is called transitive if it has an automorphism group, which is transitive on the spreads. A transitive deficiency one parallelism corresponds to a parallelism with an automorphism group which fixes one spread and is transitive on the remaining spreads.

Let $V = \{P_i\}_{i=1}^{v}$ be a finite set of *points*, and $\mathcal{B} = \{B_j\}_{j=1}^{b}$ a finite collection of k-element subsets of V, called *blocks*. $D = (V, \mathcal{B})$ is a *2-design* with parameters 2-(v,k,λ) if any 2-subset of V is contained in exactly λ blocks of $\mathcal{B}$.

A *parallel class* of a design is a partition of the point set by blocks. A *resolution* is a partition of the collection of blocks by parallel classes. The incidence of the points and t-dimensional subspaces of $PG(d, q)$ defines a 2-design (see for instance [18, 2.35-2.36]). There is a one-to-one correspondence between the parallelisms of $PG(3, 4)$ and the resolutions of the 2-(85,5,1) design of its points and lines.

Parallelisms of $PG(3, 4)$ can be obtained by Beutelspacher's general construction of parallelisms in $PG(2^n - 1, q)$ [3] , and a pair of orthogonal ones by Fuji-Hara's construction for $PG(3, q)$ [9]. All parallelisms of $PG(3, 2)$ are known. Parallelisms with predefined automorphism groups have been classified by Prince in $PG(3, 3)$ [14] and $PG(3, 5)$ [15]. Parallelisms of $PG(3, 4)$ with automorphisms of order 7 were classified by Topalova and Zhelezova [20].

There are no transitive parallelisms of $PG(3, 4)$ [6], [20]. No examples of transitive deficiency one parallelisms of $PG(3, 4)$ are known. Properties of the automorphism groups and the spreads of transitive deficiency one parallelisms of $PG(3, q)$ are derived by Biliotti, Jha, and Johnson [4], and Diaz, Johnson, and Montinaro [7], who show that the deficiency spread must be Desarguesian, and the automorphism group should contain a subgroup of order q^2.

A parallelism of $PG(3, 4)$ has 21 spreads. That is why the order of the automorphism group of a transitive deficiency one parallelism must be divisible by 20, and therefore it must have automorphisms of order 5. There are 85 points in $PG(3, 4)$. That is why a point-transitive parallelism must have an automorphism of order 5 too. We construct all (up to isomorphism) parallelisms of $PG(3, 4)$ with automorphisms of order 5. We classify them by the order of the full automorphism group. None of them is point-transitive, and none corresponds to a transitive deficiency one parallelism.

Our programmes performing the computer computations, are based on the exhaustive back track search techniques (see for instance [12, chapter 4]). To filter away isomorphic parallelisms, we find the normalizers of the subgroups of order 5 in the automorphism group of the projective space.

We use design approach to the problems. We actually make all the computations on the related to PG(3,4) designs, namely, we choose the 17 spread elements among the 357 blocks of the 2-(85,5,1) point-line design, and construct the parallelisms as its resolutions. We find a generating set of the automorphism group of $PG(3, 4)$ as generating set of the automorphism group of the related 2-(85,21,5) point-hyperplane design.

2 Construction and results

2.1 $PG(3,4)$

There are 85 points and 357 lines in $PG(3,4)$. Denote by G the full automorphism group of $PG(3,4)$. It is of order 1974067200. A spread has 17 lines which partition the point set and a parallelism has 21 spreads.

To construct $PG(3,4)$ we use $GF(4)$ with generating polynomial $x^2 = x+1$. The points of $PG(3,4)$ are then all 4-dimensional vectors (v_1, v_2, v_3, v_4) over $GF(4)$ such that if $v_k = 0$ for all $k > i$ then $v_i = 1$. We sort these 85 vectors in ascending lexicographic order and then assign them numbers such that (1,0,0,0) is number 1, and (3,3,3,1) number 85. We then construct the related designs and find the generators of their full automorphism group G.

2.2 Sylow subgroup of order 25

Since 5^2 divides the order of G, but 5^3 does not, by Sylow's Theorems (see, for instance [13, 7.2.4]) all subgroups of order 25 are conjugate, and we can choose an arbitrary one of them. We use GAP [10] to find a Sylow subgroup of order 25. We denote it G_{25}. It partitions the lines in 21 orbits, namely 13 orbits of length 25, 6 orbits of length 5, and 2 fixed lines. That is why the existence of parallelisms invariant under G_{25} is impossible. G_{25} is generated by an automorphism of order 5 without fixed points and an automorphism of order 5 with 5 fixed points. Denote the subgroups generated by each of these automorphisms by G_{5_0} and G_{5_5} respectively. We consider one by one the results about these two groups below. In the next subsection we denote by G_5 any of these two subgroups when we consider properties holding for both of them.

2.3 Construction

We sort the 357 lines (blocks of the 2-(85,5,1) design) in lexicographic order defined on the numbers of the points they contain and assign to each line a number according to this order. We construct the spreads by backtrack search. If there are already n elements in the spread, we choose the $n+1$-st one among the lines containing the first point, which is in none of the n spread elements. If a spread is fixed by the automorphism group, then if we add a line, we add the lines of its orbit too. If a spread is not fixed, we choose lines with orbits of one and the same length. Any spread we construct is lexicographically greater than the ones constructed before it.

For each spread, which is not fixed by the automorphism group, we already know the other 4 spreads of its orbit. We call the first spread *orbit leader* (a fixed spread is also an orbit leader). To obtain a parallelism we need to construct only the orbit leaders.

The rejection of isomorphic solutions is an important part of the computation. Let $\varphi \in G$. Let $\mathcal{P}_1$ be a parallelism with automorphism group $G_{\mathcal{P}_1}$, and let $\mathcal{P}_2 = \varphi\mathcal{P}_1$. Denote by $G_{\mathcal{P}_2}$ the automorphism group of the parallelism $\mathcal{P}_2$. Let $\alpha \in G_{\mathcal{P}_1}$ and $\beta \in G_{\mathcal{P}_2}$. Then $\beta\varphi\mathcal{P}_1 = \varphi\alpha\mathcal{P}_1$ and thus $\beta = \varphi\alpha\varphi^{-1}$ and $G_{\mathcal{P}_2} = \varphi G_{\mathcal{P}_1}\varphi^{-1}$. To filter away isomorphic solutions we want to check if there is some permutation $\varphi \in G$ such that it maps a parallelism with G_5 to another parallelism with G_5. Thus we obtain $G_5 = \varphi G_5 \varphi^{-1}$ and therefore we are interested in the normalizer $N(G_5)$ of G_5 in G, which is defined as $N(G_5) = \{g \in G \mid gG_5g^{-1} = G_5\}$. If an automorphism $\varphi \in G$ transforms one of the constructed parallelisms with G_5 into another one, then $\varphi \in N(G_5)$. For each parallelism we obtain, we check if an automorphism of $N(G_5)$ transforms it into a parallelism with a lexicographically smaller orbit leader sequence, and drop it if so.

We obtain G_{25} and the normalizers of G_{5_0} and G_{5_5} by GAP [10]. The other computer results are obtained by our own C++ programs. Most of them are checked by two different programmes (algorithms) developed independently by the two authors.

2.4 Automorphism of order 5 without fixed points

The group G_{5_0} fixes 17 lines. The remaining 340 lines form 68 orbits of length 5. Two types of parallelisms invariant under G_{5_0} are possible:

1. Type 1: Parallelisms with
   - one fixed spread made of the 17 fixed lines and
   - four orbits of five spreads each
2. Type 2: Parallelisms with
   - 1 fixed spread with 7 fixed lines and 2 orbits of 5 lines and
   - 5 fixed spreads with 2 fixed lines and 3 orbits of 5 lines and
   - three orbits of five spreads each

The normalizer $N(G_{5_0})$ is a group of order 81600. Since this order is divisible by 85, point-transitive parallelisms can be expected. We obtain 176 parallelisms of type 1, and 170 of type 2. The results are summarized in Table 1, where all the different orders of the full automorphism groups of the obtained parallelisms are presented in column *Aut*. In the next three columns we give the number of parallelisms with this order of the full automorphism group, which are obtained with G_{5_0} and of type 1, with G_{5_0} and of type 2, and with G_{5_5}.

2.5 Automorphism of order 5 with 5 fixed points

The group G_{5_5} fixes 2 lines. The remaining 355 lines form 71 orbits of length 5. The parallelisms invariant under G_{5_5} have:

- one fixed spread containing 2 fixed lines and 3 orbits of 5 lines and
- four orbits of five spreads each

The normalizer $N(G_{5_5})$ is a group of order 3600. We construct 27924 parallelisms (Table 1). The full automorphism groups of order 20 partition the spreads in 4 orbits, while those of order 60 - in 3 orbits, so these parallelisms do not correspond to transitive deficiency one parallelisms.

2.6 Summary

Table 1: The order of the full automorphism group of the parallelisms

Aut \ G_5	G_{5_0} type 1	G_{5_0} type 2	G_{5_5}	All
5	156	170	27836	28162
10	8		40	48
15	4		12	16
20			30	30
30	8		2	10
60			4	4
All	176	170	27924	28270

The classification results show that there are no transitive deficiency one parallelisms and no point-transitive parallelisms in $PG(3,4)$.

References

[1] E. F. Jr. Assmus, J. D. Key, *Designs and their Codes, Cambridge Tracts in Mathematics*, **103**, Cambridge University Press, 1992.

[2] Th. Beth, D. Jungnickel, H. Lenz, *Design Theory*, Cambridge University Press, 1993.

[3] A. Beutelspacher, On parallelisms in finite projective spaces, *Geometriae Dedicata*, **3** (1), 35–45, 1974.

[4] M. Biliotti, V. Jha, N. L. Johnson, Classification of transitive deficiency one partial parallelisms, *Bull. Belg. Math. Soc.*, **12**, 371–391, 2005.

[5] C. J. Colbourn and J. H. Dinitz (Eds.), *The CRC Handbook of Combinatorial Designs*, CRC Press, 2006, second edition.

[6] R. Denniston, Some packings of projective spaces, *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.*, **52** (8), 36–40, 1972.

[7] E. Diaz, N. L. Johnson, A. Montinaro, Transitive deficiency one partial parallelisms, *Advances and Applications in Discrete Mathematics*, **1** (1), 1–34, 2008.

[8] J. Eisfeld and L. Storme, (Partial) t-spreads and minimal t-covers in finite projective spaces, in *Lecture notes from the Socrates Intensive Course on Finite Geometry and its Applications*, Ghent, April 2000.

[9] Fuji-Hara, Mutually 2-orthogonal resolutions of finite projective space, *Ars Combinatoria*, **21**, 163–166, 1986.

[10] GAP - Groups, Algorithms, Programming - a System for Computational Discrete Algebra (http://www.gap-system.org/).

[11] N. L. Johnson, Parallelisms of projective spaces, *Journal of Geometry* **76**, 110–182, 2003.

[12] P. Kaski and P. Östergård, *Classification algorithms for codes and designs*, Springer, Berlin, 2006.

[13] W. Ledermann, A. J. Weir, Group Theory, *Longman*, 1996, second edition.

[14] A. R. Prince, Parallelisms of PG(3,3) invariant under a collineation of order 5, in *Mostly Finite Geometries* (Iowa City, 1996) (N. L. Johnson, ed.), *Lecture Notes in Pure and Applied Mathematics*, **190**, Marcel Dekker, New York, 383–390, l997.

[15] A. R. Prince, The cyclic parallelisms of PG(3,5), *European Journal of Combinatorics* **19**,(5), 613–616, 1998.

[16] N. Silberstein, Coding Theory and Projective Spaces, PhD thesis, Israel Institute of Technology, Haifa, 2011.

[17] N. Silberstein, T. Etzion, Codes and designs related to lifted MRD codes , *IEEE International Symposium on Information Theory Proceedings (ISIT)*,St. Petersburg, ISSN: 2157-8095, 2288–2292, 2011 .

[18] L. Storme, Finite Geometry, *The CRC Handbook of Combinatorial Designs*, CRC Press, 702–729, 2006, second edition.

[19] V. D. Tonchev, *Combinatorial configurations, Longman Scientific and Technical*, New York, 1988.

[20] S. Topalova, S. Zhelezova, Parallelisms of PG(3,4) with automorphisms of order 7, *Proceedings of the Sixth International Workshop on Optimal Codes and Related Topics, Varna, 2009, Bulgaria*, 184–188, 2009.