The nonexistence of transitive 2-parallelisms of $PG(5,3)$

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Abstract. 2-spread is a set of 2-dimensional subspaces of $PG(d,q)$, which partition the point set. A 2-parallelism is a partition of the set of 2-dimensional subspaces by 2-spreads. Johnson and Montinaro in their paper "The transitive t-parallelisms of a finite projective space" point out that the existence of transitive 2-parallelisms of $PG(5,3)$ is an open question. In the present paper we establish that there are no transitive 2-parallelisms of $PG(5,3)$.

1 Introduction

For the basic concepts and notations concerning projective spaces, spreads and parallelisms, refer, for instance, to [5], [8].

Definition 1.1 *A t-spread in $PG(d,q)$ is a set of t-dimensional subspaces which partition the point set.*

Definition 1.2 *A t-parallelism of $PG(d,q)$ is a partition of the set of all t-dimensional subspaces by t-spreads.*

A necessary condition for the existence of t-spreads and t-parallelisms in $PG(d,q)$ is $t+1 \mid d+1$ [5].

Definition 1.3 *t-parallelisms with an automorphism group, which is transitive on the t-spreads are called transitive [6].*

Definition 1.4 *Two t-parallelisms are isomorphic if there is an automorphism of $PG(d,q)$ mapping the first one to the second.*

A construction of 1-parallelisms in $PG(d,2)$ is presented by Zaicev, Zinoviev and Semakov [15] and independently by Baker [1], and in $PG(2^n-1,q)$ by Beutelspacher [2]. Many constructions are known in $PG(3,q)$ due to Denniston [3], Johnson [4], [5], Johnson and Pomareda [7], Penttila and Williams [9], Prince [10] and [12].

Examples of transitive 1-parallelisms of $PG(3,q)$ are presented in [3], [9] and [12], and of $PG(5,2)$ in [13]. The first examples of transitive 2-parallelisms are constructed in [14].

In [6, Theorem 1] Johnson and Montinaro show that transitive t-parallelisms of $PG(n, q)$ exist only when $t = 1$, or when $t = 2$ and $(n, q) = (5, 2), (5, 3)$. They determine the automorphism group of $PG(5, 3)$ which can act transitively on the 2-spreads.

In the present work we use this result to establish the nonexistence of transitive 2-parallelisms of $PG(5, 3)$.

2 Investigation and result

There are 364 points, 11011 lines and 33880 planes in $PG(5, 3)$, each line is incident with 4 points and each plane with 13 points. A 2-spread has 28 planes which partition the point set and a 2-parallelism has 1210 2-spreads.

Let $\mathcal{P}$ be a 2-parallelism of $PG(5, 3)$ and let G_i be an automorphism group of $PG(5, 3)$ which leaves $\mathcal{P}$ invariant and acts transitively on it. The second part of the main Theorem in [6] claims that G_i is $Z_{242}.Z_5$. So if there are any transitive 2-parallelisms of $PG(5, 3)$ their full authomorphism group has to be of order $i = 1210$. Denote it by G_{1210}.

Since 11^2 divides the order of the full automorphism group G of $PG(5, 3)$, but 11^3 does not, by Sylow Theorems all subgroups of order 11^2 are conjugate, and we can choose an arbitrary one of them - G_{121}.

We found G_{1210} as the normalizer of G_{121} in G by using the computer system for algebraic computations GAP (http://www.gap-system.org/). G_{1210} is generated by one of the nontrivial automorphisms of order 121, an automorphism of order 2 and an automorphism of order 5.

G_{1210} partitions the 33880 planes of $PG(5, 3)$ in two orbits of lenght 605 and 27 orbits of lenght 1210. We can obtain a transitive 2-parallelism if by each permutaion of G_{1210} each plane is mapped to a different $PG(5, 3)$ plane. This is impossible for the planes of the shorter orbits. So we can conclude that transitive 2-parallelisms of $PG(5, 3)$ do not exist.

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