Search for a moving target in a graph

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The Princess Problem

The Princess

A princess lives in a row of seventeen adjacent rooms, each connected by a door to each room next to it. Each room also has a door to the outside. The princess enjoys the rooms but never stays in the same room two days in a row: at the end of each day she moves from the room she occupied to one of the rooms next to it (she chooses randomly).

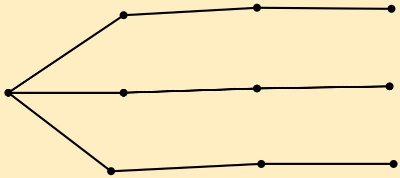
The Princess Problem

The Prince

On the first of June a prince arrives from a faraway kingdom to woo the princess. The princesss guardian explains the habits of the princess and the rules he must follow: Each day he may knock on a single outside door. If the princess is behind it she will open it and meet the prince. If not, the prince gets another chance the next day. Unfortunately the prince must return to his kingdom on July 1. Can he devise a strategy to make sure he meets the princess before then?

The soldier and the sniper

Fortification system consists of bunkers. Some of the bunkers are connected with trenches as shown on the picture.



A soldier is hiding in one of the bunkers. A sniper shoots on a bunkers. Between two consecutive shots the soldier necessarily runs across one of the trenches in a nearby bunker.
Do there exist a strategy for the sniper that makes sure he hits the soldier?

k-chase

A Princess occupies a vertex of a given graph G and a Suitor is trying to find her.

On each turn, the Suitor examines k vertices of G looking for the Princess (and, if he finds her, the game ends).

Following this, the Princess moves to an adjacent vertex of G and the turn is complete.

Chase depth

The *chase depth* of a given graph G is the minimum positive integer k such that the Suitor has a winning strategy for the k-chase on G.

Research problems

- Given a positive integer k, describe all graphs G of chase depth k.
- Given a graph G, find its chase depth.
- Given a graph G of chase depth k, find the minimum number of turns necessary for the Suitor to win.

Results

- For $k = 1$, we give a complete characterization of graphs for which it is possible for the Suitor to find the Princess.
- We find the minimum k for which the Suitor finds the Princess when G is a rectangular grid of size $2n \times 2n$.

Definitions

Let $S \subset V(G)$. Denote by $c(S)$ the set of all direct successors of the vertices in S. Given a strategy for the Suitor, we write S_i for the set of vertices that the Princess could occupy at the beginning of turn i and Q_i for the set of vertices investigated by the Suitor on turn i.

So, $S_1 = V(G)$ and for all i we have $S_{i+1} = c(S_i \setminus Q_i)$. Moreover, a k-chase is a win for the Suitor if and only if for some i he can achieve $|S_i| \leq k$.

The *stratification* of a connected graph G is the finest partitioning $O_1, O_2, \ldots, O_m$ of $V(G)$ such that $c(O_i) = O_{i+1}$ for all i (where $O_{m+1} = O_1$). The O_i are the *strata* of G.

Stratification lemma

The chase depth of a graph G with stratification $O_1, O_2, \dots, O_m$ does not change if, at the beginning of the game, the Suitor is given the additional information that the Princess occupies a vertex in O_1.

Expansion lemma

Let G be a graph with stratification O_1, O_2, ..., O_m and chase depth k.
Suppose that there exist positive integers l, l_1, l_2, ..., l_m such that, for all i, $l_i < |O_i|$ and every subset S of O_i of size at least l_i satisfies $|c(S)| \geq l + l_{i+1}$ (where $l_{m+1} = l_1$).
Then $k \geq l + 1$.

Let $O_1, O_2, \dots, O_m$ be the stratification of an undirected graph G. Then $m = 2$ if G is bipartite and $m = 1$ otherwise.

Any path has chase depth 1

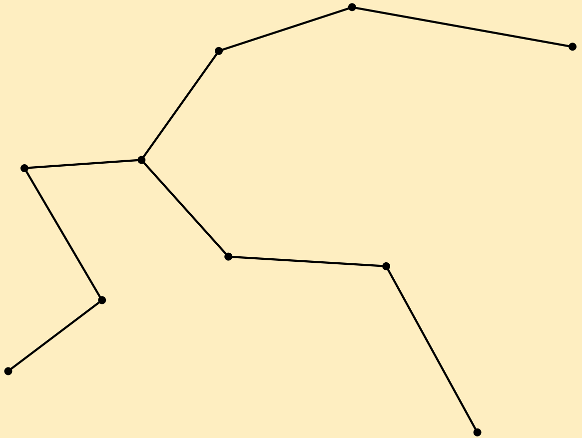
Such a graph is bipartite with strata $O_1 = \{v_1, v_3, \dots\}$ and $O_2 = \{v_2, v_4, \dots\}$.

By the stratification lemma, we may assume that the Princess is in O_2.
The Suitor finds her by successively looking into vertices v_2, v_3, $\dots$, v_{n-1}.

Remark. For a path of length n and $k = 1$, the minimum number of turns necessary for the Suitor to win is $2n - 2$.

Any cycle has chase depth 2

The graph $G^\star$ with vertices $\{u_i \mid 0 \le i \le 9\}$ and edges $\{(u_0, u_i) \mid i = 1, 2, 3\} \cup \{(u_{3k+i-3}, u_{3k+i}) \mid k = 1, 2; i = 1, 2, 3\}$ has chase depth 2.



Graphs of chase depth 1

A graph G has chase depth 1 if and only if G is acyclic and $G^\star \not\subset G$.

The chase depth of a $2n \times 2n$ grid equals $n + 1$.

An $m \times n$ grid is isomorphic to an $m \times n$ chessboard with two squares being adjacent iff they share a common side.

- Exhibit a winning $(n + 1)$-width O_1-chase strategy for the Suitor.
- Every set S of n^2 white squares satisfies $|c(S)| \geq n^2 + n$.