CHARACTERIZATION OF HIGHLY DIVISIBLE ARCS

Assia Rousseva
Sofia University

(joint work with Ivan Landjev)

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1. $(t \mod q)$-Arcs

◇ A multiset in $\mathrm{PG}(r,q)$ is a mapping

$$\mathcal{K} : \begin{cases} \mathcal{P} & \rightarrow & \mathbb{N}_0, \\ P & \rightarrow & \mathcal{K}(P). \end{cases}$$

◇ $\mathcal{K}(P)$ – multiplicity of the point P.

◇ $\mathcal{Q} \subset \mathcal{P}$: $\mathcal{K}(\mathcal{Q}) = \sum_{P \in \mathcal{Q}} \mathcal{K}(P)$ – multiplicity of the set $\mathcal{Q}$.

◇ $\mathcal{K}(\mathcal{P})$ – the cardinality of $\mathcal{K}$.

◇ a_i – the number of hyperplanes H with $\mathcal{K}(H) = i$

◇ $(a_i)_{i \geq 0}$ – the spectrum of $\mathcal{K}$

Definition. (n,w)-arc in $\mathrm{PG}(r,q)$: a multiset $\mathcal{K}$ with

1) $\mathcal{K}(\mathcal{P}) = n$;

2) for every hyperplane H: $\mathcal{K}(H) \le w$;

3) there exists a hyperplane H_0: $\mathcal{K}(H_0) = w$.

Definition. (n,w)-blocking set in $\mathrm{PG}(r,q)$

(or (n,w)-minihyper): a multiset $\mathcal{K}$ with

1) $\mathcal{K}(\mathcal{P}) = n$;

2) for every hyperplane H: $\mathcal{K}(H) \ge w$;

3) there exists a hyperplane H_0: $\mathcal{K}(H_0) = w$.

Definition. Let $t < q$ be a positive integer.

An arc $\mathcal{F}$ in $\mathrm{PG}(r, q)$ is called a $(t \mod q)$-arc if

(1) every point P has multiplicity at most t: $\mathcal{F}(P) \leq t$;

(2) every subspace S of positive dimension has multiplicity

$\mathcal{F}(S) \equiv t \pmod{q}$.

Remark. It is enough to require (2) only for the lines.

2. General Constructions for $(t \mod q)$-Arcs

Theorem A. Let $t_1 < q$ and $t_2 < q$ be positive integers. The sum of a $(t_1 \mod q)$-arc and a $(t_2 \mod q)$-arc in $\mathrm{PG}(r,q)$ is a $(t \mod q)$-arc with $t = t_1 + t_2 \pmod q$ provided the multiplicities of all points do not exceed t. In particular, the sum of t hyperplanes in $\mathrm{PG}(r,q)$ is a $(t \mod q)$-arc.

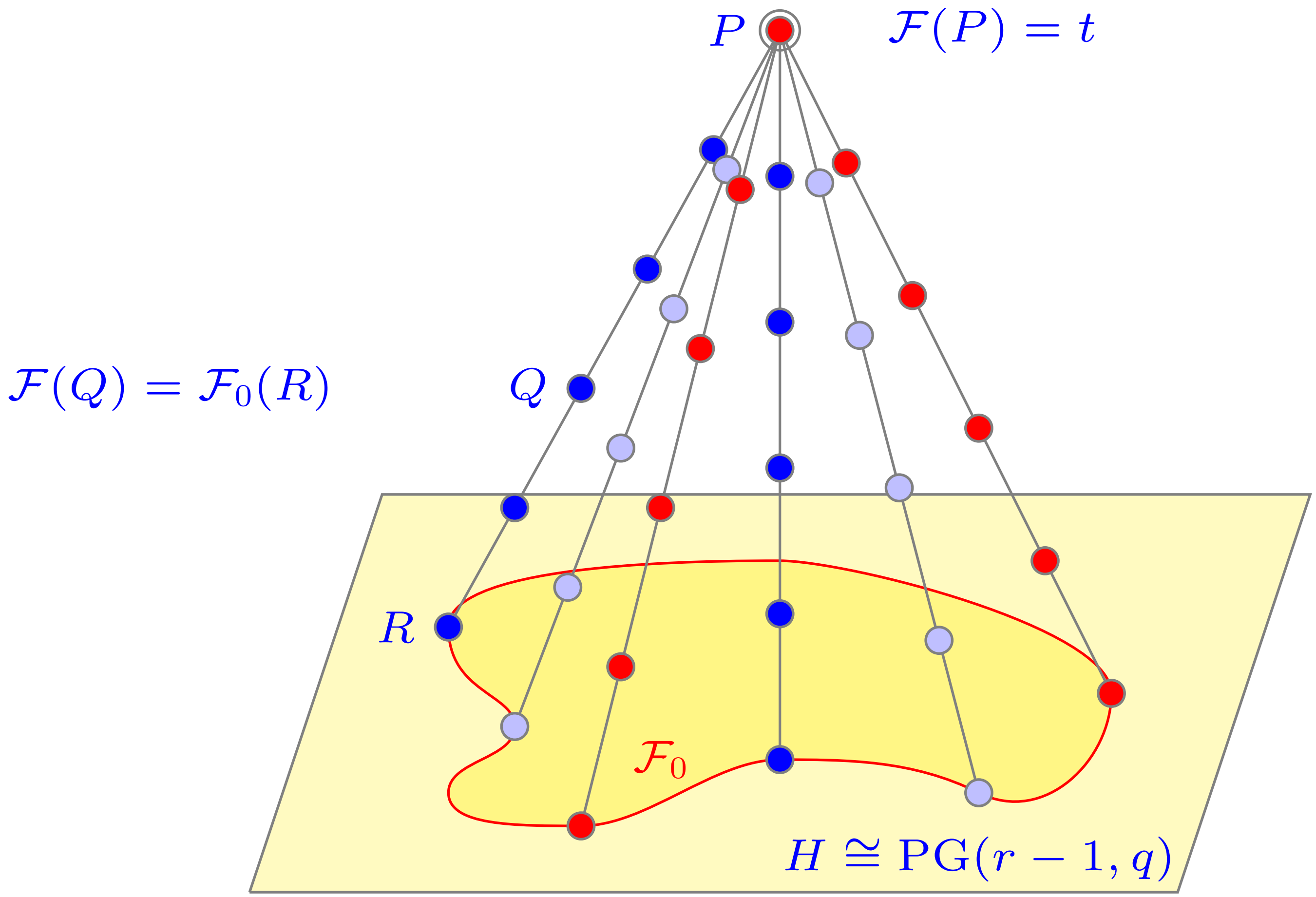
Theorem B. Let $\mathcal{F}_0$ be a $(t \mod q)$-arc in a hyperplane $H \cong \mathrm{PG}(r-1,q)$. of $\Sigma = \mathrm{PG}(r,q)$. For a fixed point $P \in \Sigma \setminus H$, define an arc $\mathcal{F}$ in Σ as follows:

– $\mathcal{F}(P) = t$;

– for each point $Q \neq P$: $\mathcal{F}(Q) = \mathcal{F}_0(R)$ where $R = \langle P, Q \rangle \cap H$.

Then the arc $\mathcal{F}$ is a $(t \mod q)$-arc in $\mathrm{PG}(r,q)$ of size $q|\mathcal{F}_0| + t$.

Definition. $(t \mod q)$-arcs obtained by Theorem B are called lifted arcs.


P
$\mathcal{F}(P) = t$
$\mathcal{F}(Q) = \mathcal{F}_0(R)$
Q
R
$\mathcal{F}_0$
$H \cong \mathrm{PG}(r-1, q)$

Lemma. Let a $(t \mod q)$-arc $\mathcal{F}$ be lifted from the points P and Q, $P \neq Q$. Then $\mathcal{F}$ is also lifted from any point on the line PQ. In particular, the lifting points of a $\mathcal{F}$ form a subspace.

Theorem C. Let $\mathcal{F}$ be a $(t \mod q)$-arc in $\mathrm{PG}(r,q)$ such that the restriction $\mathcal{F}|_H$ to every hyperplane H of $\mathrm{PG}(r,q)$ is lifted. Then $\mathcal{F}$ is also a lifted arc.

Corollary. If all $(t \mod q)$-arcs in $\mathrm{PG}(r_0,q)$ are lifted then so are all $(t \mod q)$-arcs in $\mathrm{PG}(r,q)$ for all $r \geq r_0$.

$\mathcal{F}$: an arc in $\Sigma = \mathrm{PG}(r, q)$

$\mathcal{H}$ – the set of all hyperplanes in Σ

σ - a function such that $\sigma(\mathcal{F}(H))$ is a non-negative integer for all $H \in \mathcal{H}$.

The arc $\mathcal{F}^\sigma$ in $\widetilde{\Sigma}$

$$\mathcal{F}^\sigma : \begin{cases} \mathcal{H} & \rightarrow & \mathbb{N}_0 \\ H & \rightarrow & \sigma(\mathcal{F}(H)) \end{cases}$$

is called the σ-dual of $\mathcal{F}$.

Theorem D. The arc $\mathcal{F}$ is a $(t \mod q)$-arc in $\mathrm{PG}(2, q)$ of size $mq + t$ if and only if the arc $\mathcal{F}^\sigma$ with $\sigma(x) = (x - t)/q$ is an $((m - t)q + m, m - t)$-blocking set in the dual plane with line multiplicities $m - t, m - t + 1, \ldots, m$.

3. $(1 \mod q)$ and $(2 \mod q)$-Arcs

An $(1 \mod q)$-arc in $\mathrm{PG}(r, q)$ is either a hyperplane, or the complete space.

A $(2 \mod q)$-arc in $\mathrm{PG}(2, q)$, q odd, is one of the following (Maruta, 2003)

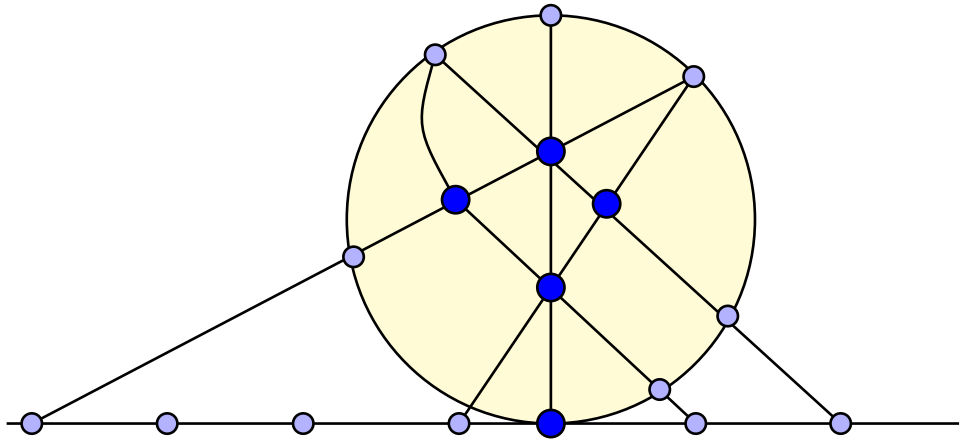
(1) A lifted arc from a 2-line.

(2) A lifted arc from a $(q + 2)$-line.

(3) A lifted arc from a $(2q + 2)$-line.

(4) An exceptional $(2 \mod q)$-arc: an oval plus a tangent plus twice all internal points of the oval.

(4) The exceptional $(2 \mod q)$-arc



- 2-points
- 1-points

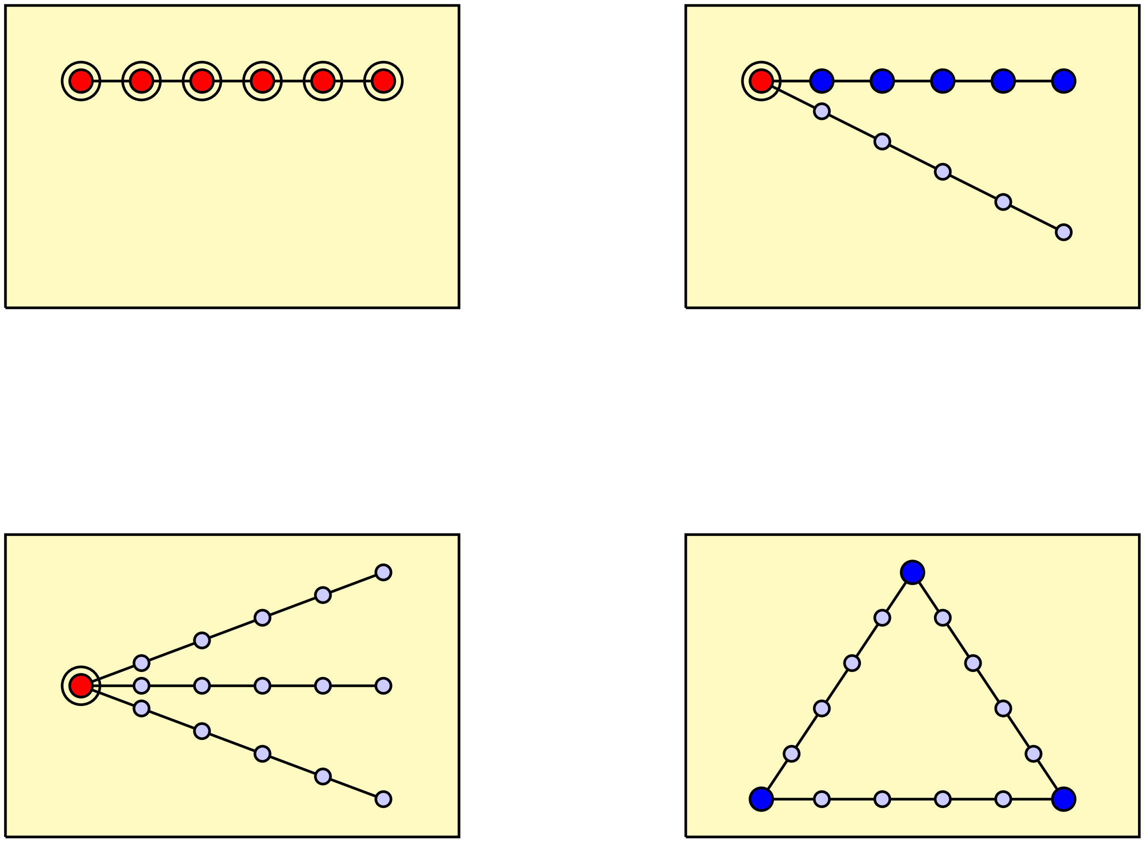
Lemma. Let $\mathcal{F}$ be a $(2 \mod q)$-arc in $\mathrm{PG}(3,q)$, q odd. Let there exist a plane π such that the restriction $\mathcal{F}|_\pi$ is of type (4). Then $\mathcal{F}$ is a lifted arc.

Theorem E. Every $(2 \mod q)$-arc in $\mathrm{PG}(r,q)$, $r \geq 3$, q odd, is lifted.

Corollary. Every $(2 \mod q)$-arc in $\mathrm{PG}(r,q)$, $r \geq 3$, q odd, contains a hyperplane in its support.

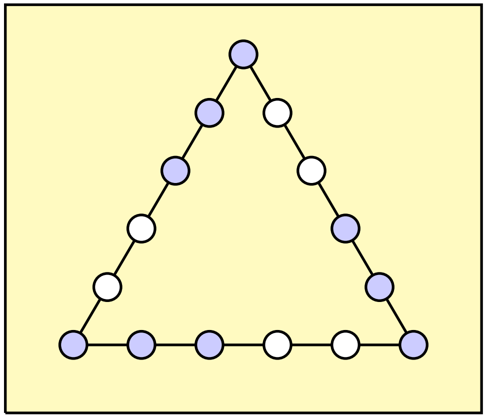
4. $(3 \mod 5)$-arcs in $\mathrm{PG}(2,5)$

$(18, \{3, 8, 13, 18\})$-arcs



$\mathcal{F}$: $(23, \{3, 8\})$-**arc**

$\mathcal{F}^\sigma$:



$\mathcal{F}^\sigma$: (9,1)-blocking set

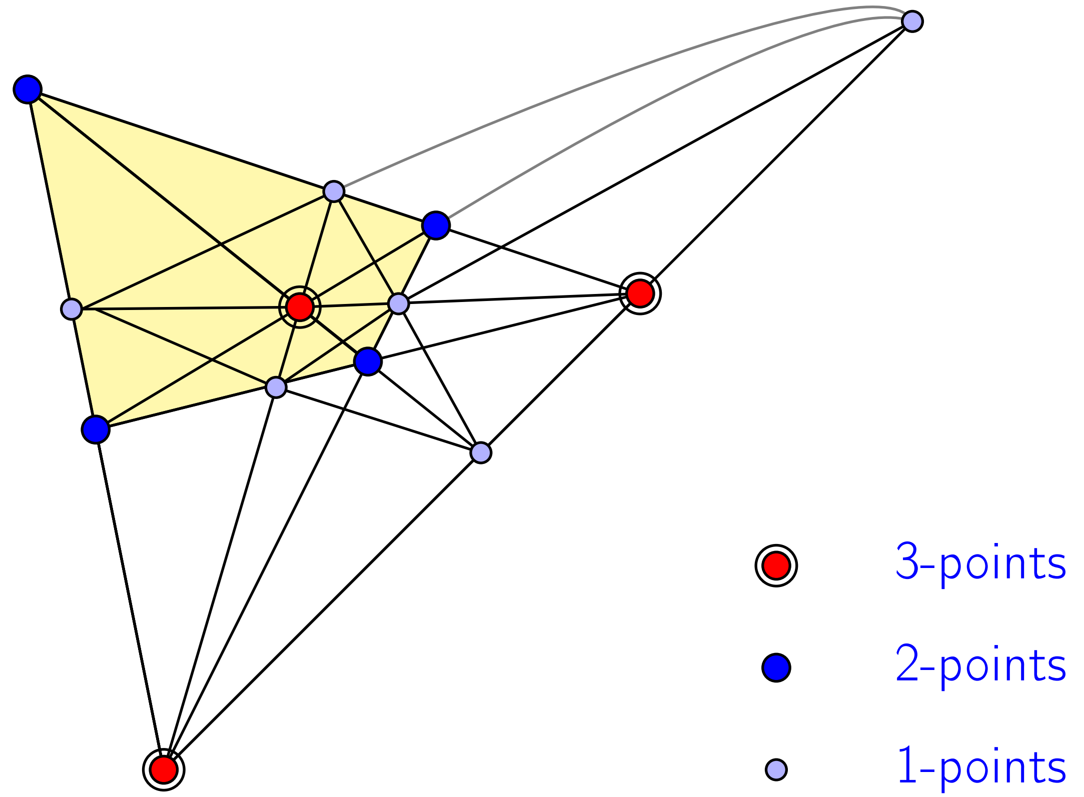
with line multiplicities $1, 2, 3, 4$

$\mathcal{F}$: $(28, \{3, 8\})$-**arc**

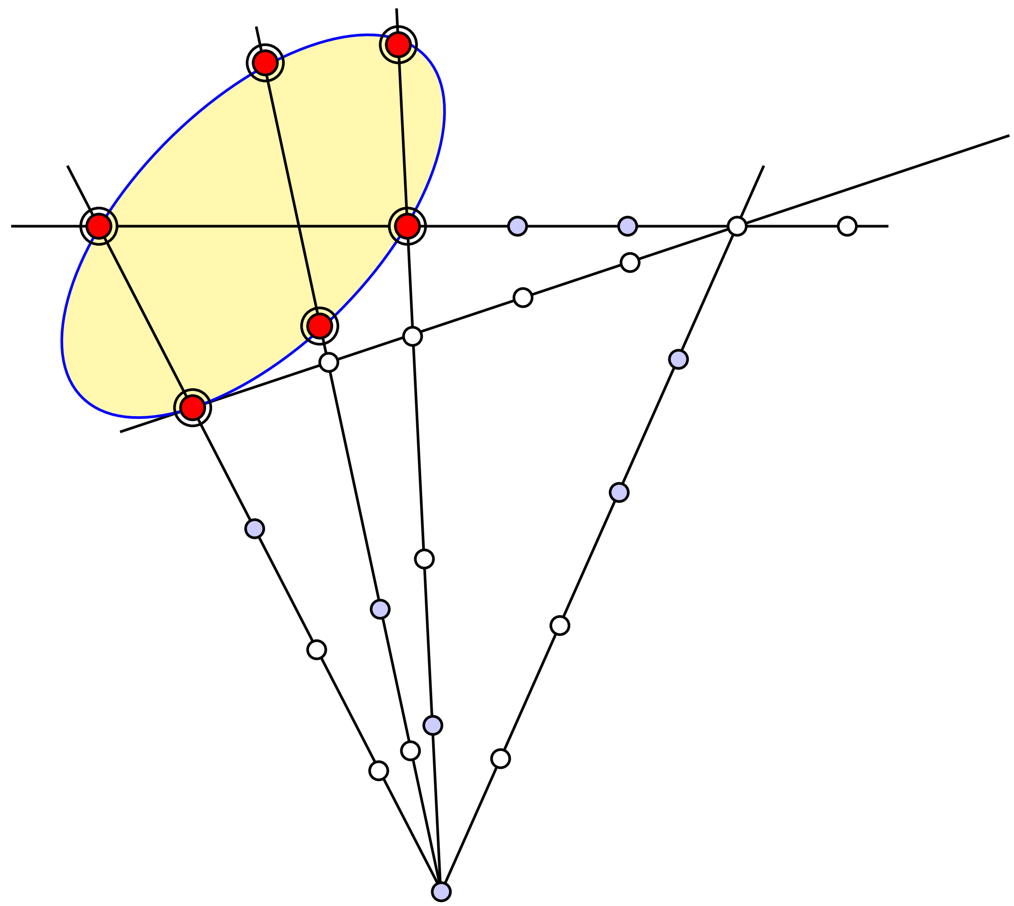
$\mathcal{F}^\sigma$: $(15, 2)$-blocking set with line multiplicities $2, 3, 4, 5$

$\mathcal{F}^\sigma$: the complement of the unique $(16, 4)$-arc without external lines

The $(23, \{3, 8\})$-arc



The $(28, \{3, 8\})$-arc



$\mathcal{F}$: $(33, \{3, 8, 13\})$-**arc**

$\mathcal{F}^\sigma$: $(21, 3)$-blocking set with line multiplicities $3, 4, 5, 6$

$\mathcal{F}^\sigma$ is one of the following:

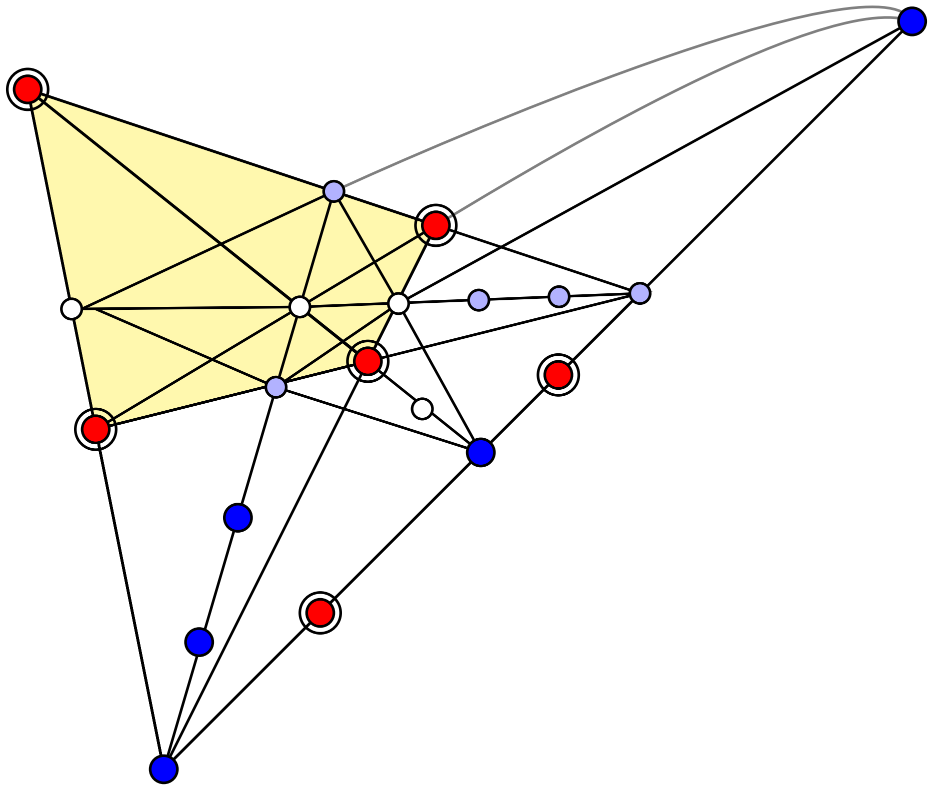
(1) the complement of the seven non-isomorphic $(10, 3)$-arcs; $\Lambda_2 = 0$

(2) the complement of the $(11, 3)$-arc with four external lines; a point not on an external line is doubled; $\Lambda_2 = 1$

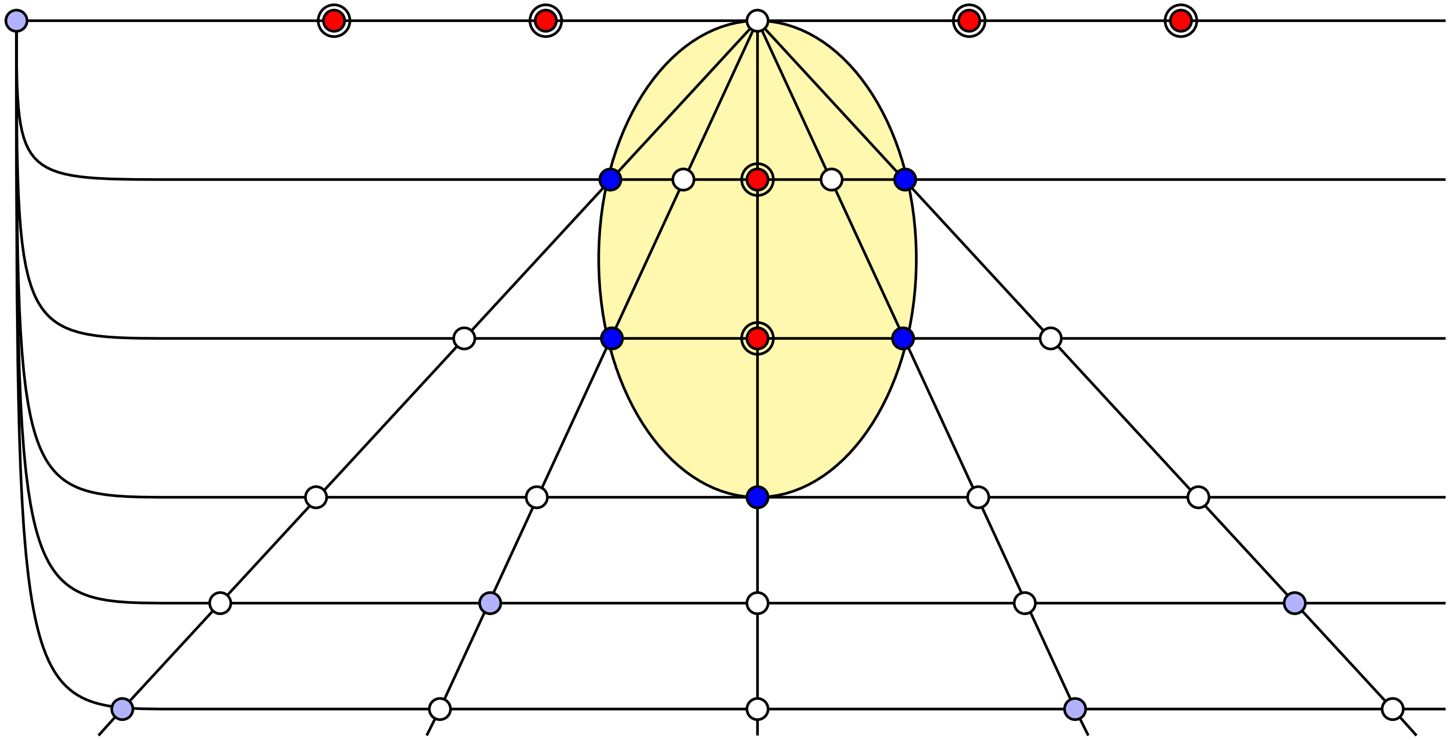
(3) one double point which forms an oval with five of the 0-points; the tangent in the 2-point is a 3-line; $\Lambda_2 = 1$

(4) $\mathrm{PG}(2, 5)$ minus a triangle with vertices of multiplicity $2, 2, 1$; $\Lambda_2 = 2$

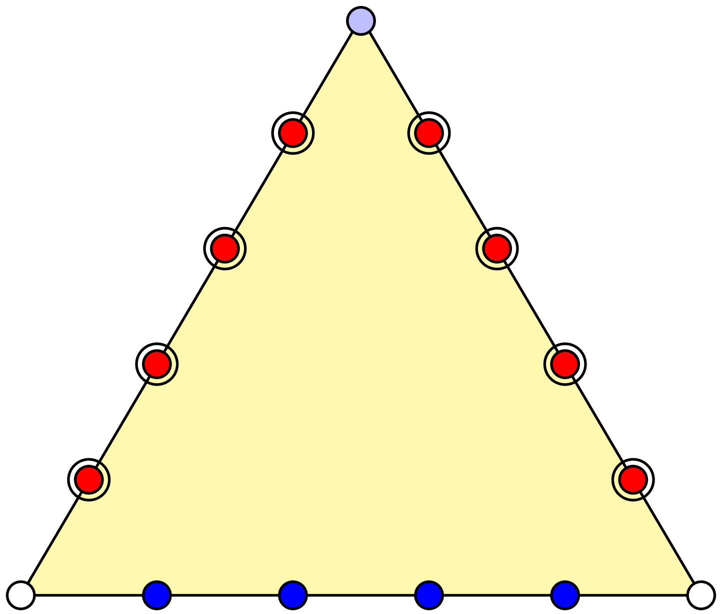
(2) The first $(33, \{3, 8, 13\})$-arc with one 13-line



(3) the second $(33, \{3, 8, 13\})$-arc with one 13-line



(4) $(33, \{3, 8, 13\})$-arc with two 13-lines



$\mathcal{F}$: $(38, \{3, 8, 13\})$**-arc**

$\mathcal{F}^\sigma$: $(27, 4)$-blocking set with line multiplicities $4, 5, 6, 7$

There exist at least twenty non-equivalent $(3 \mod 5)$-arcs of size 38.

Theorem F. Every $(3 \mod 5)$-arc $\mathcal{F}$ in $\mathrm{PG}(3,5)$ with $|\mathcal{F}| \leq 168$ is a lifted arc (obtained by Theorem B). In particular, $|\mathcal{F}| = 93, 118, 143$, or 168.

Conjecture. A $(t \mod q)$-arc in $\mathrm{PG}(r,q)$, $r \geq 3$, is a lifted arc or the sum of lifted arcs.

Remark. We have no example of a $(t \mod q)$-arc in $\mathrm{PG}(r,q)$, $r \geq 3$ which is not the sum of lifted arcs.

5. Recent Developments

Definition. Let $0 \leq t < q$ be an integer.

An arc $\mathcal{F}$ in $\mathrm{PG}(r, q)$ is called a $(t \mod q)$-arc if

(1) every point P has multiplicity at most $q - 1$:

$\mathcal{F}(P) \leq q - 1$;

(2) every subspace S of positive dimension has multiplicity

$\mathcal{F}(S) \equiv t \pmod{q}$.

Lemma. Every $(0 \mod p)$-arc in $\mathrm{PG}(3,p)$, p prime, is a sum (over $\mathbb{F}_p$) of at most p lifted arcs.

Sketch of proof.

A – the points-by-lines incidence matrix of $\mathrm{PG}(3,p)$, (Hamada, 1968)

$$\mathrm{rk}_p\, A = \frac{1}{6}(5p^3 + 3p^2 + 4p + 6)$$

Each $(0 \mod p)$-arc is represented as a vector:

$$\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{F}_p^n,\ n = p^3 + p^2 + p + 1,$$

$$\boldsymbol{x}A = \mathbf{0}.$$

The $(0 \mod p)$-arcs form a vector space of dimension

$$\frac{1}{6}(p^3+3p^2+2p)=\binom{p+2}{3}.$$

On the other hand:

$P_1, P_2, \ldots, P_p$ – points in general position

V_i – the vector space of $(0 \mod p)$-arcs lifted from P_i

$$\dim V_i = \binom{p+1}{2}, \ \dim V_i \cap V_j = p, \ \dim V_i \cap V_j \cap V_k = 1.$$

$$\dim(V_1 + \ldots + V_p) \quad = \quad \frac{1}{6}(p^3 + 3p^2 + 2p).$$

Theorem G. Every $(t \mod p)$-arc in $\mathrm{PG}(r, p)$, $r \geq 1$, is a sum of lifted arcs (over $\mathbb{F}_p$).