A Data Structure for Dynamic Segment Intersection

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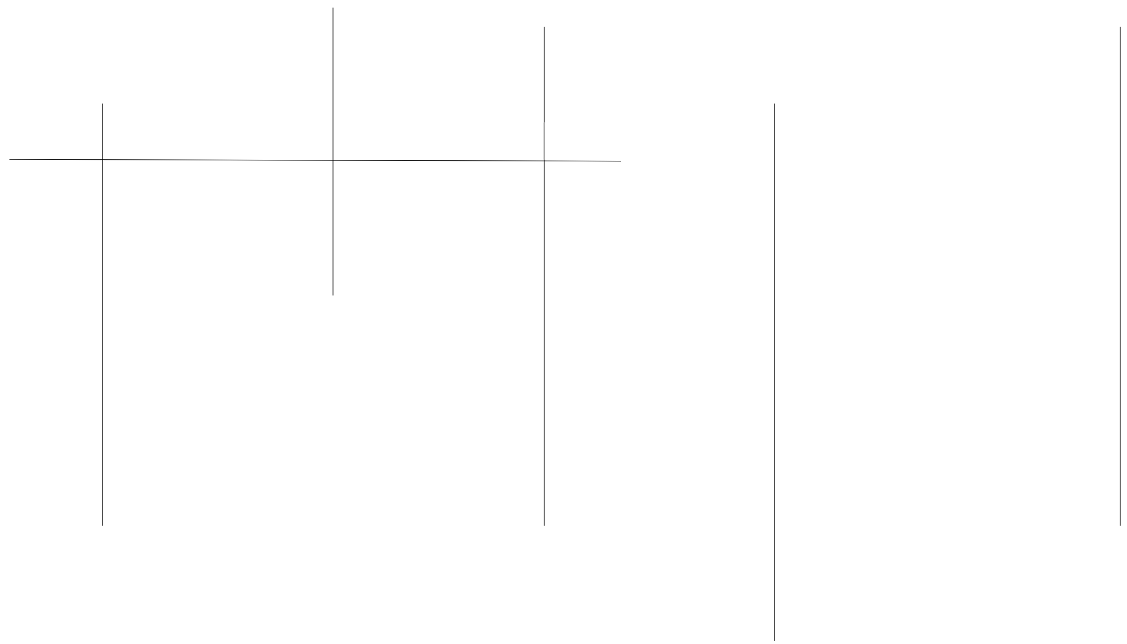
Problem Statement
The Segment Intersection Problem

Given a set of axis-aligned segments in the plane, support the following operations



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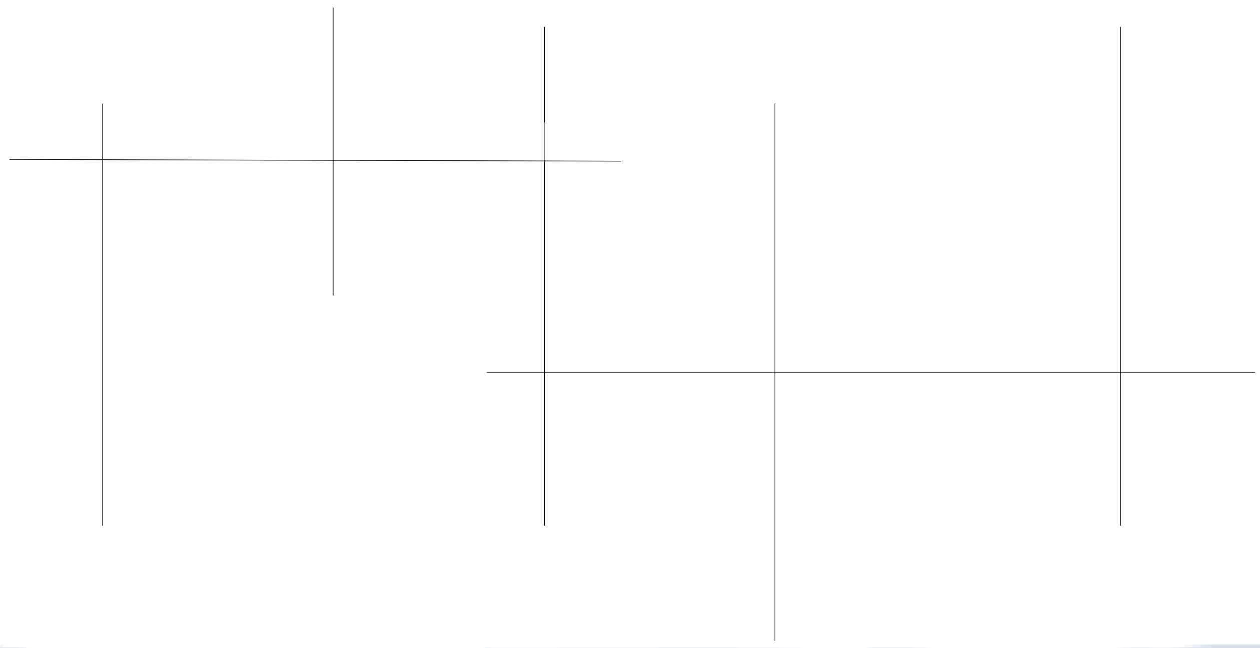

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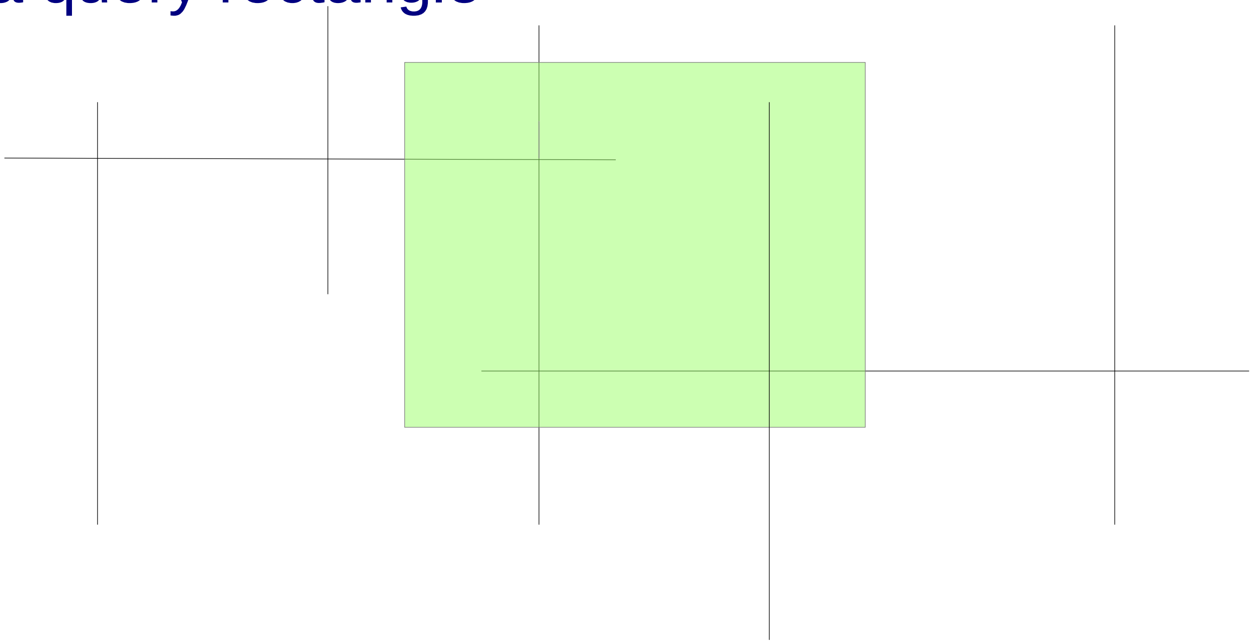
Given a set of axis-aligned segments in the plane, support the following operations

- **Insertion**: insert a new segment
- **Query**: report all pairwise intersection points of segments inside a query rectangle



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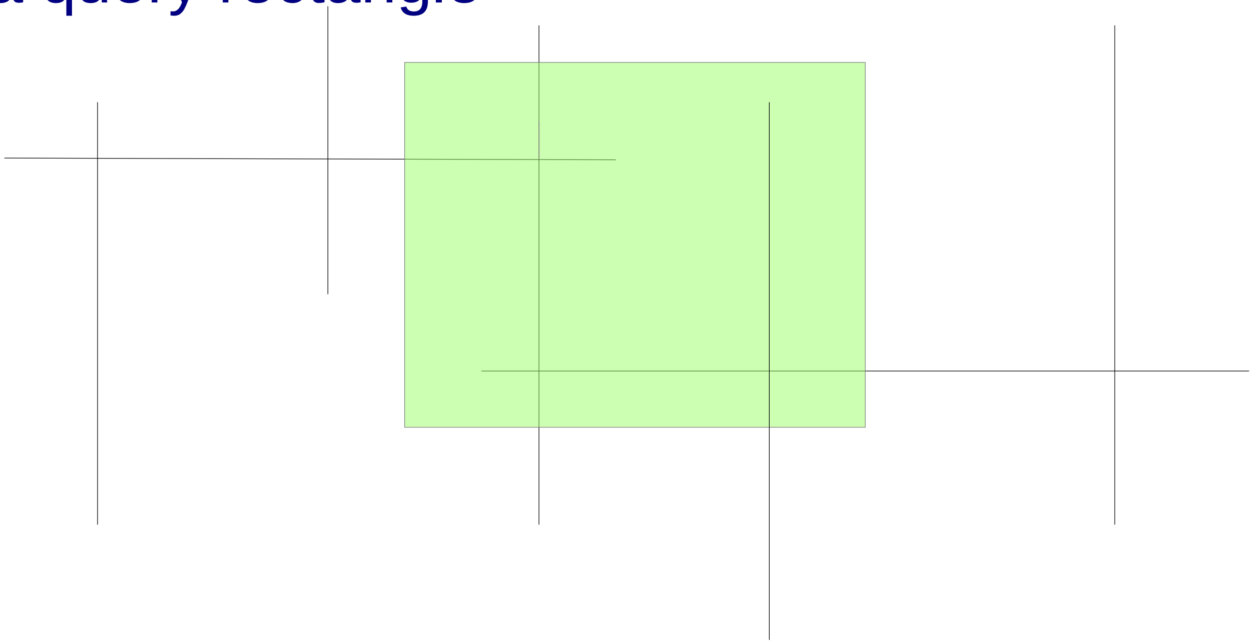

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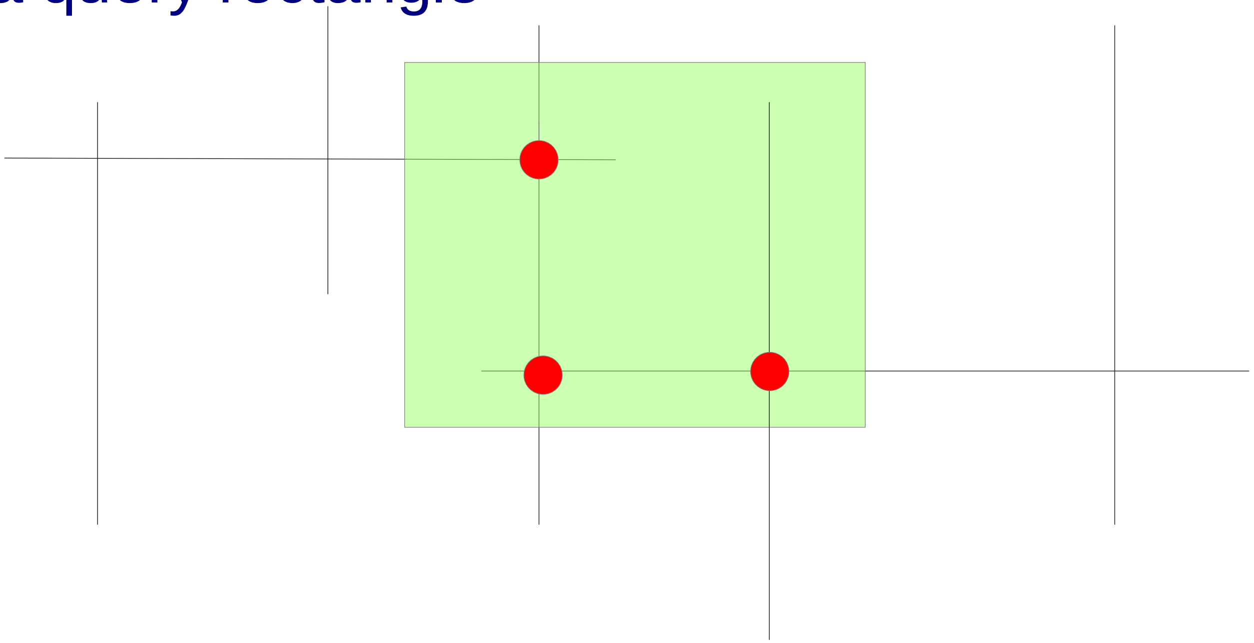


Desired properties

- Small time complexity of insertion
- Small time complexity of query
- Small preprocessing time
- Small space complexity

Related work
Static version of the problem

Finding Pairwise Intersections Inside a Query Range

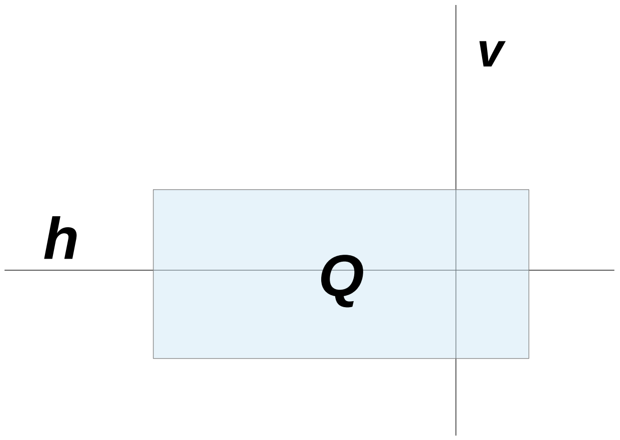
- by Mark de Berg, Joachim Gudmundsson, and Ali D. Mehrabi
- Supports only queries, not insertions

Outline

- Problem statement
- Static version of the problem
- Reduction of the dynamic version to a different problem
- Solution
- Results
- Conclusion

Approach for the static case

- Construct a set $O^*(Q)$:
  - If $h \cap v \in Q$, $h \in O^*(Q)$ or $v \in O^*(Q)$.
- For each $s \in O^*(Q)$:
  - Find all segments t: $s \cap Q \cap t \neq \emptyset$ using a multidimensional range tree.



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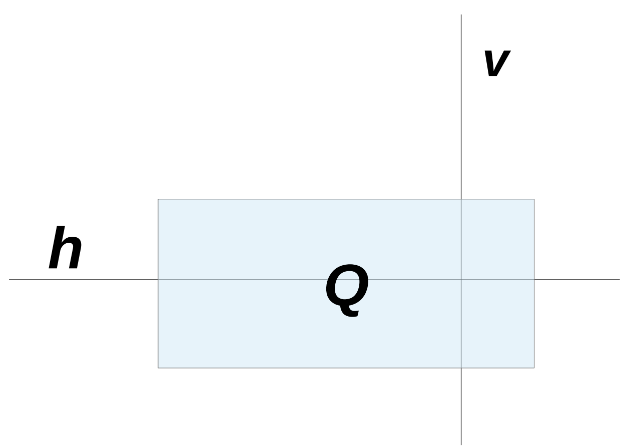
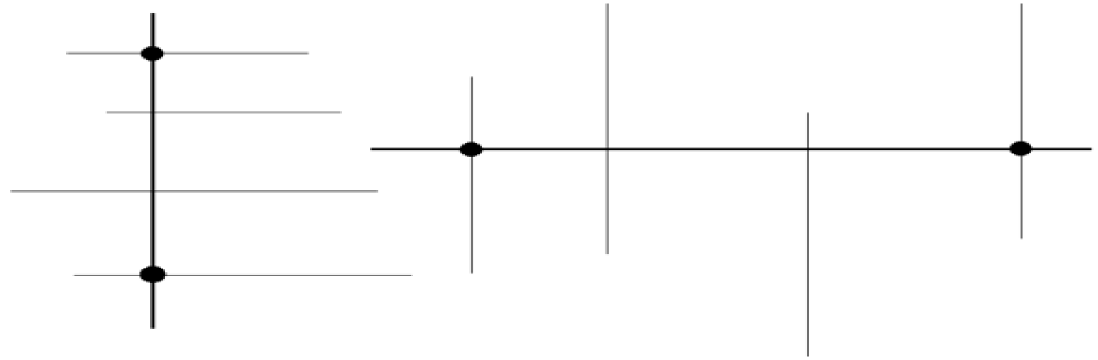
Easy: See [de Berg, Gudmundsson, and Mehrabi, 2015]



Approach for the static case

Witness points W:

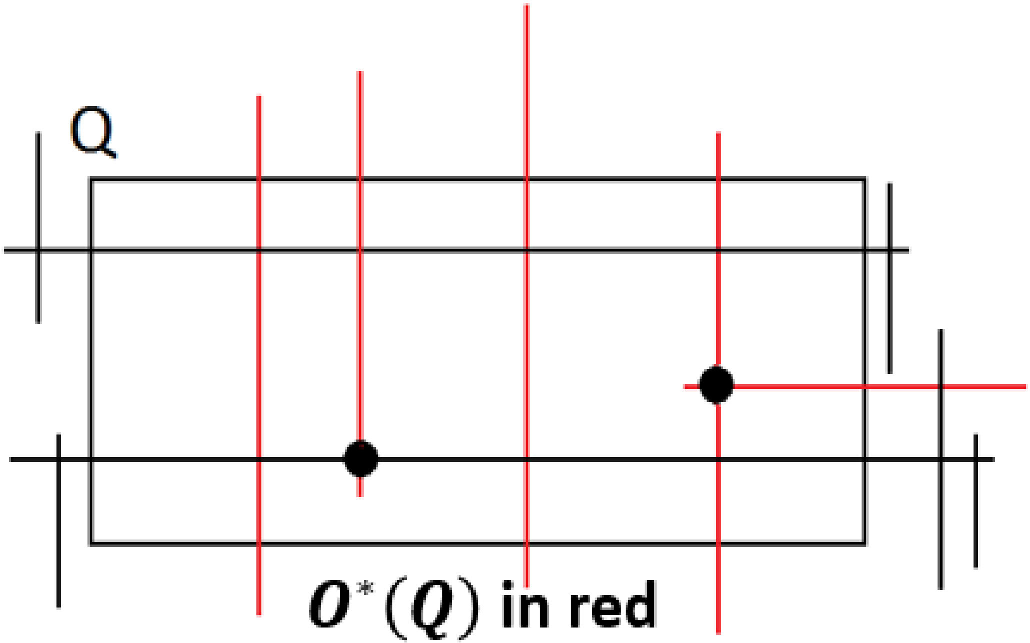
- $\forall$ v**, the topmost and bottommost intersection points of** v**,** $w_1, w_2 \in W$ $(s_{w_1} = v, s_{w_2} = v)$
- $\forall$ h**, the leftmost and rightmost intersection points of** h**,** $w_1, w_2 \in W$ $(s_{w_1} = h, s_{w_2} = h)$



Approach for the static case

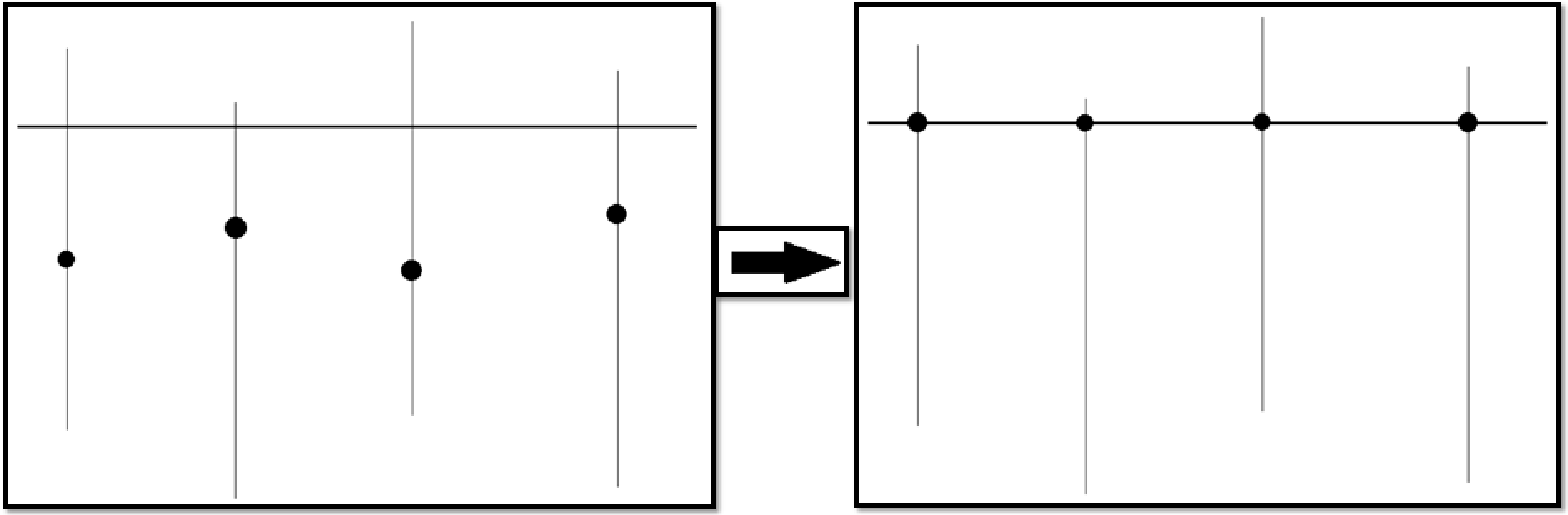
$O^*(Q)$ contains:

- $\{s_w | w \in Q, w \in W\}$ (each segment that has an associated witness point in Q)
- All segments that intersect Q entirely from top to bottom
  (only if there are segments that intersect Q entirely from left to right)



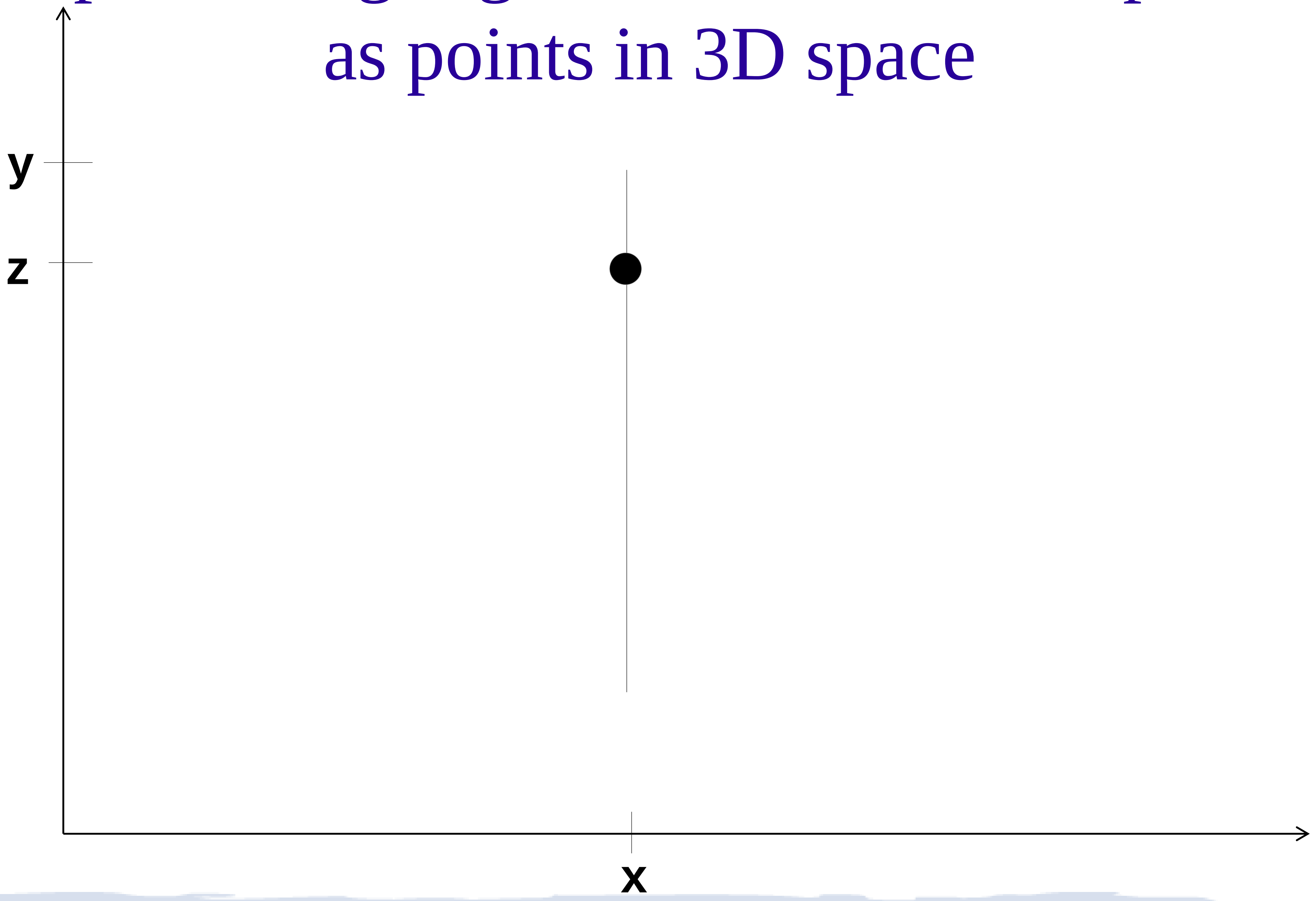
$O^*(Q)$ in red

Approach for the dynamic case
Insertion of a new horizontal segment

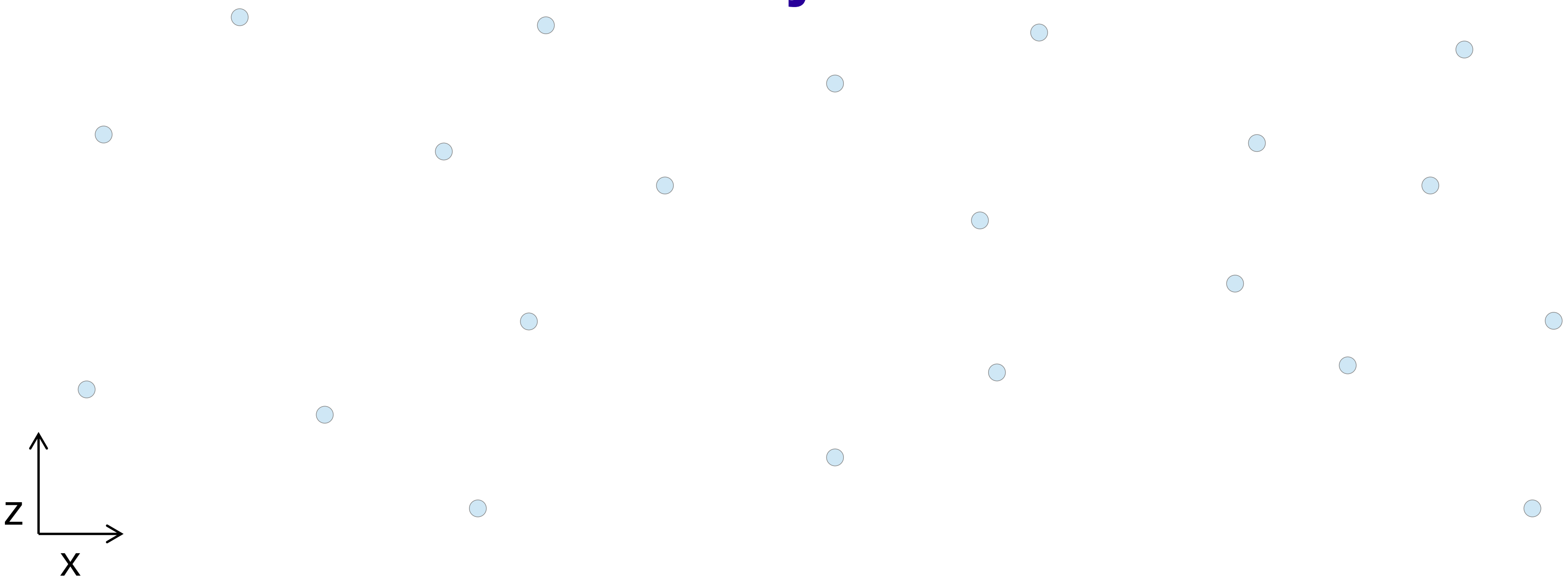


How the witness points of vertical segments change as a result of the insertion of a new horizontal segment

Representing segments with witness points as points in 3D space



The Core of the Problem
The Point Projection Problem



Support the following operations on a set of points in 3D space:

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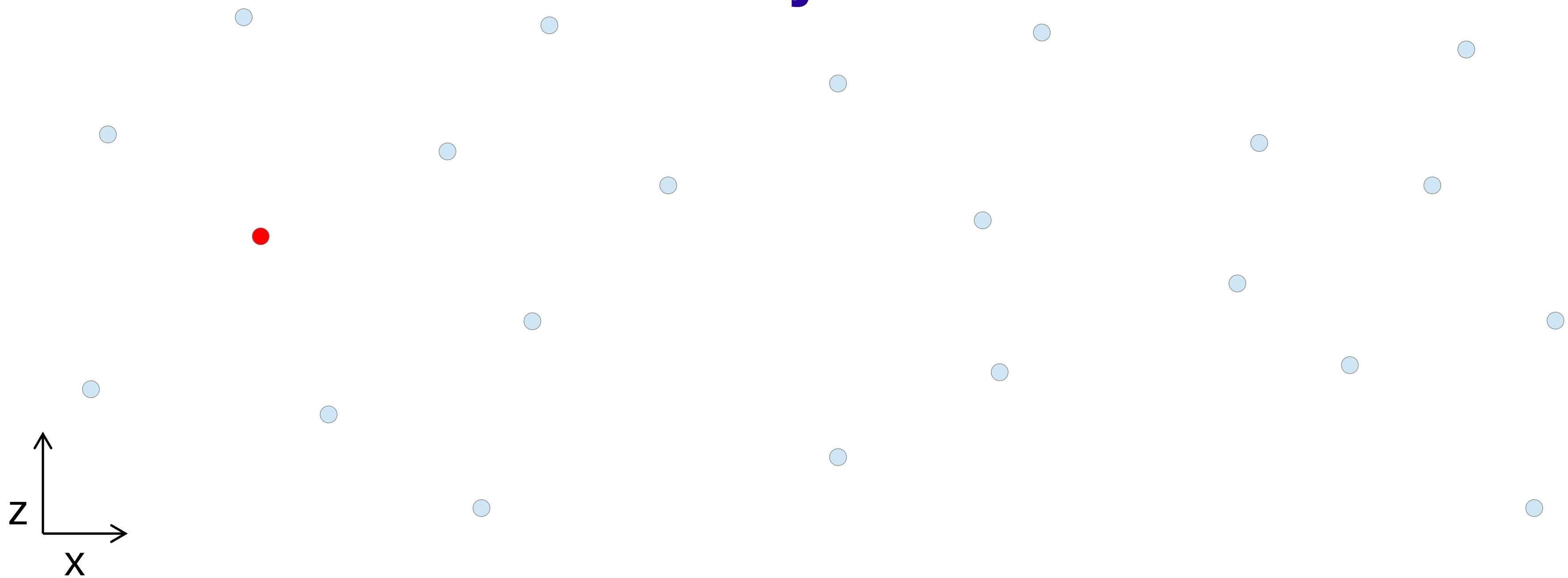

Support the following operations on a set of points in 3D space:

- Insert a point (x, y, z). **(insertion)**

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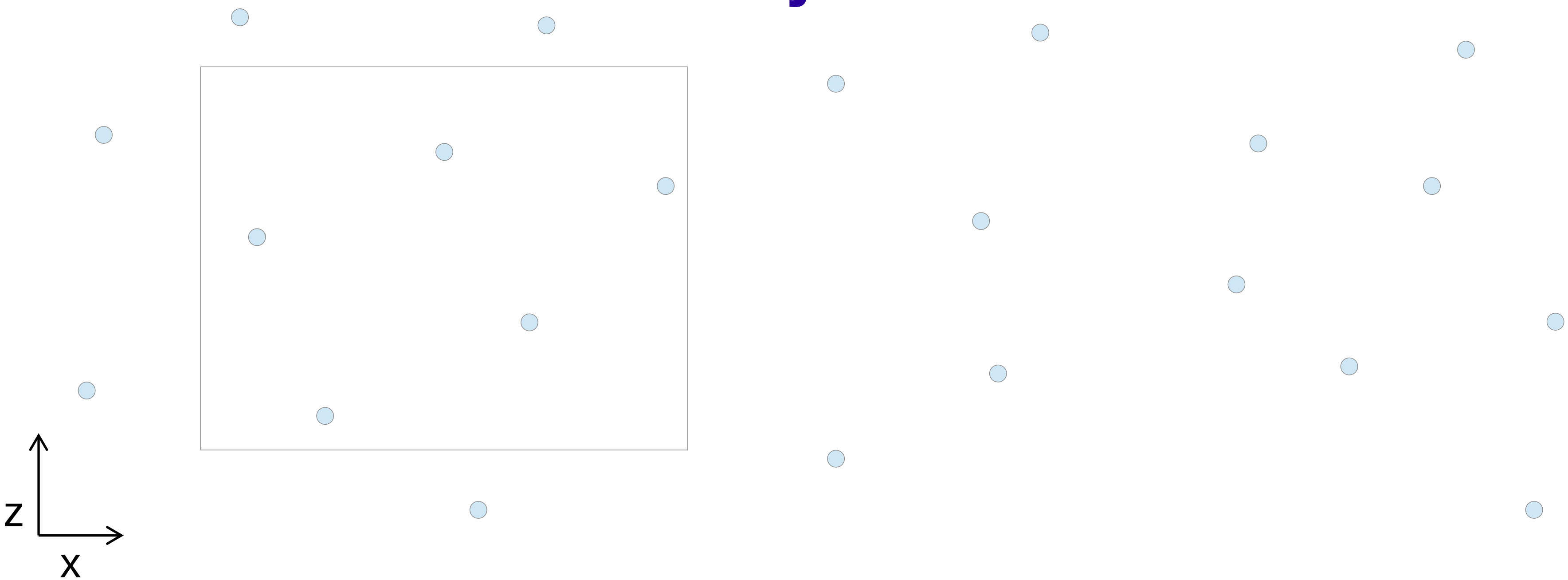

Support the following operations on a set of points in 3D space:

- Insert a point (x, y, z).
- Report all points in the box $[x_1, x_2] \times (-\infty, \infty) \times [z_1, z_2]$. **(query)**

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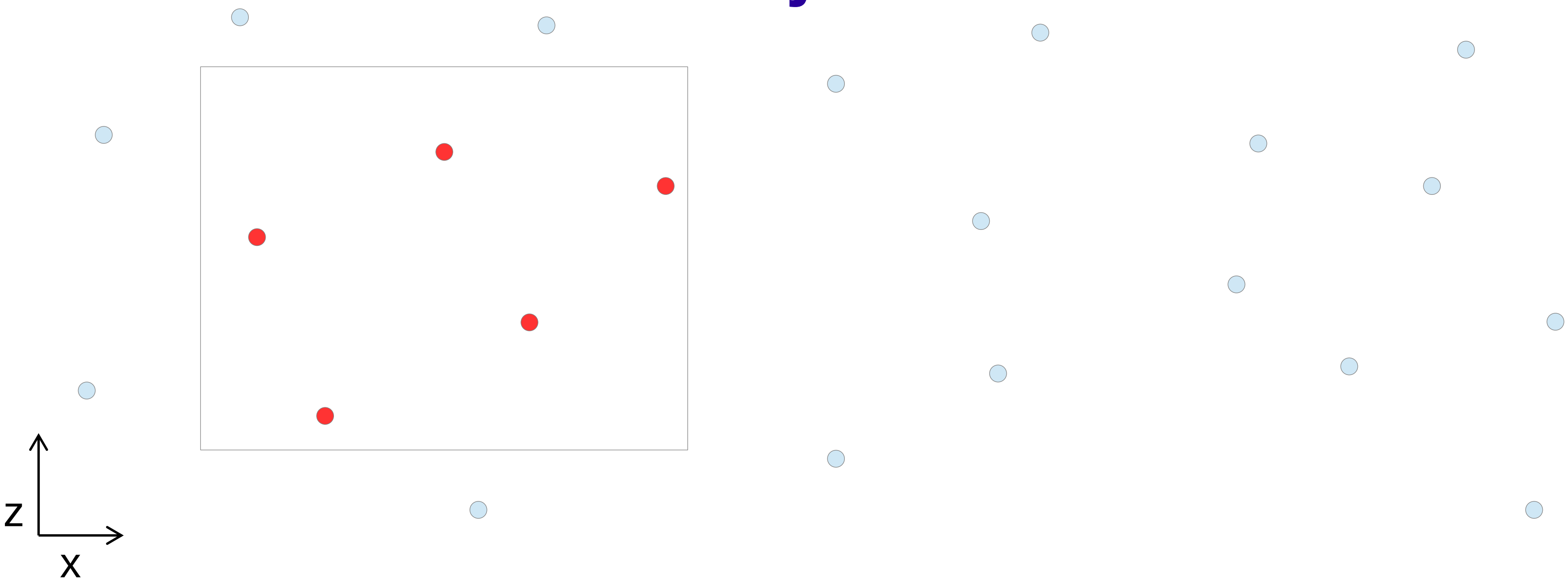

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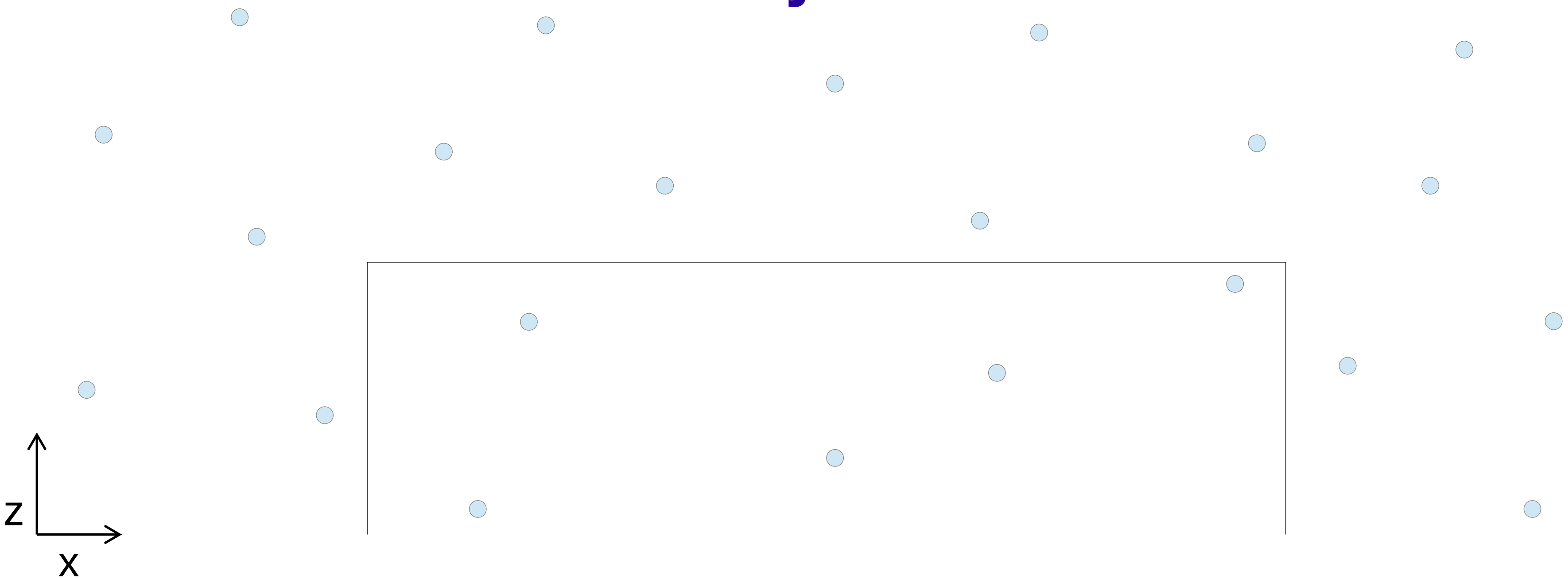

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- For all points (x, y, z) in the box $[x_1, x_2] \times (y_1, \infty) \times [-\infty, z_2]$, change z to z_2.

(update)

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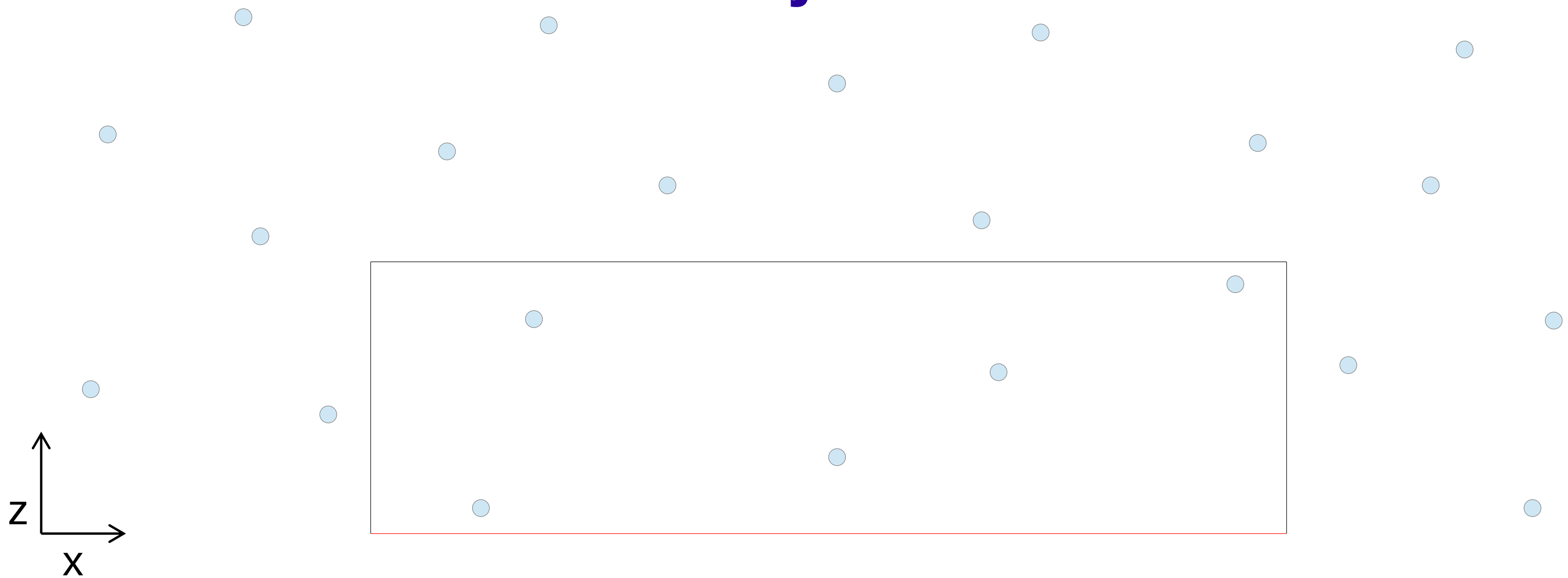

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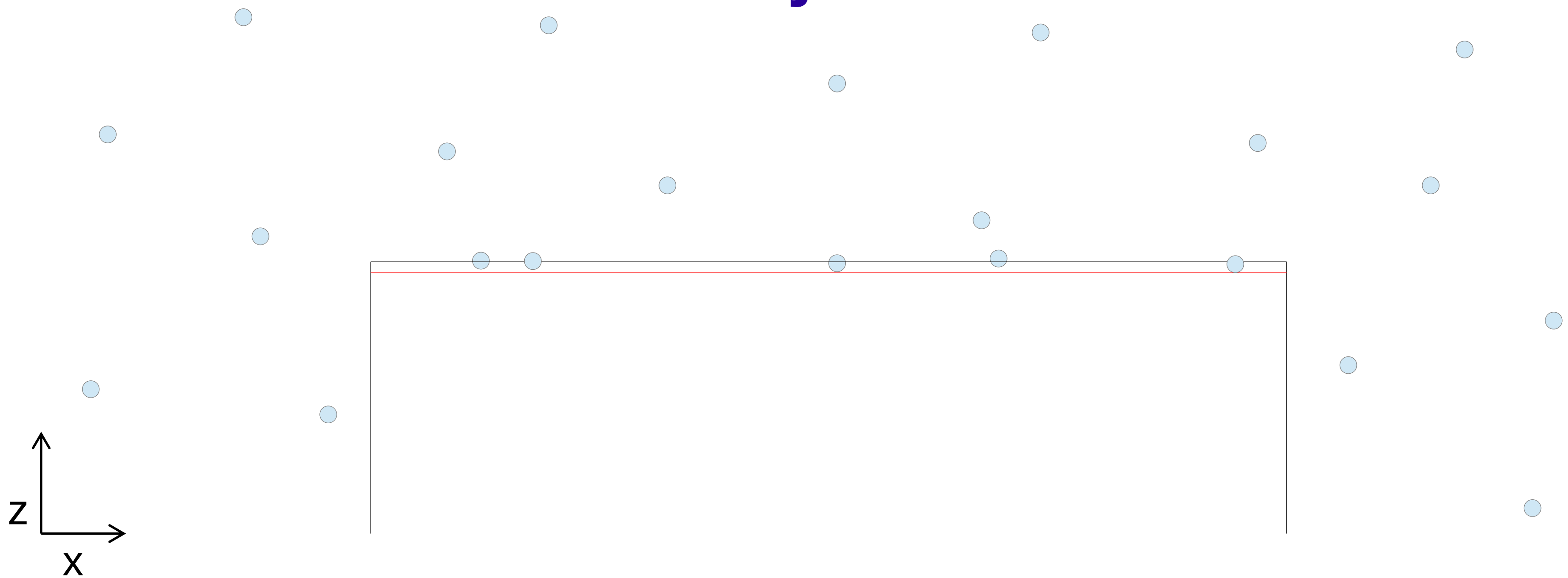

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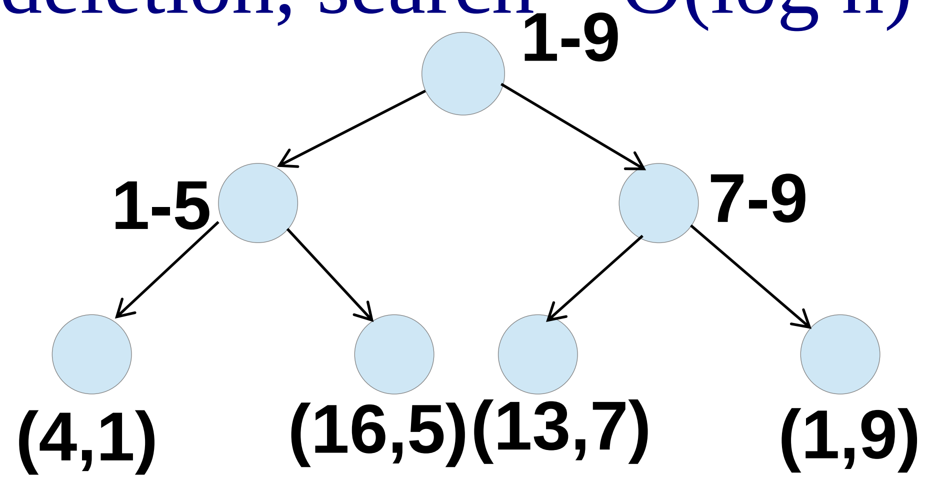
Data Structure

- K-d tree
- AVL range trees [Lamoureux, 1995]
- The binary static to dynamic transformation [Saxe and Bentley, 1979]
- Potential function analysis

AVL range tree

- Space-partitioning data structure for organizing points by one of their dimensions
- Dynamic, balanced
- Insertion, deletion, search – O(log n)

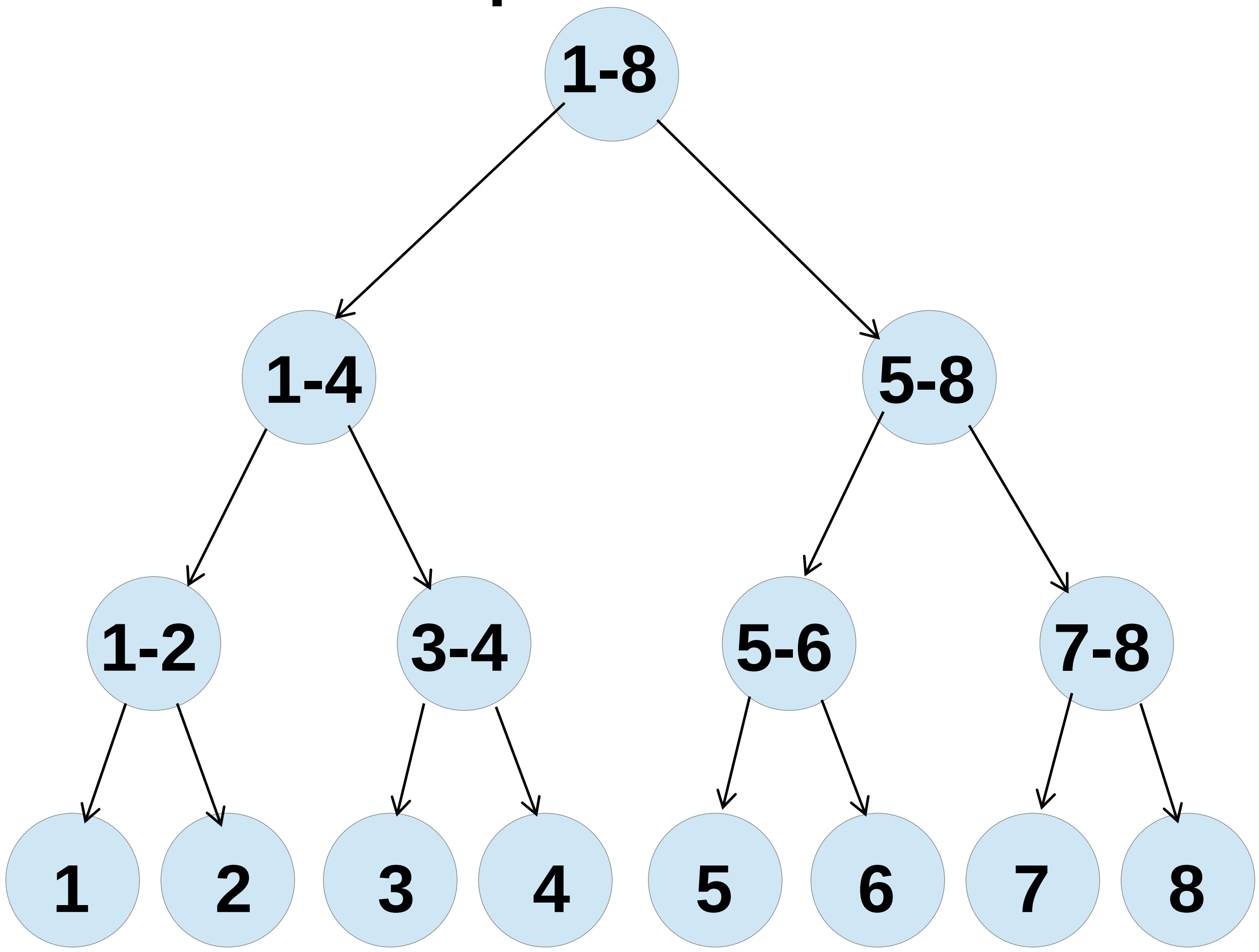


A range tree organized by the y coordinate

AVL range tree - an important fact

We can represent any interval using at most 2 log n nodes of the tree.

Example: interval 2-6



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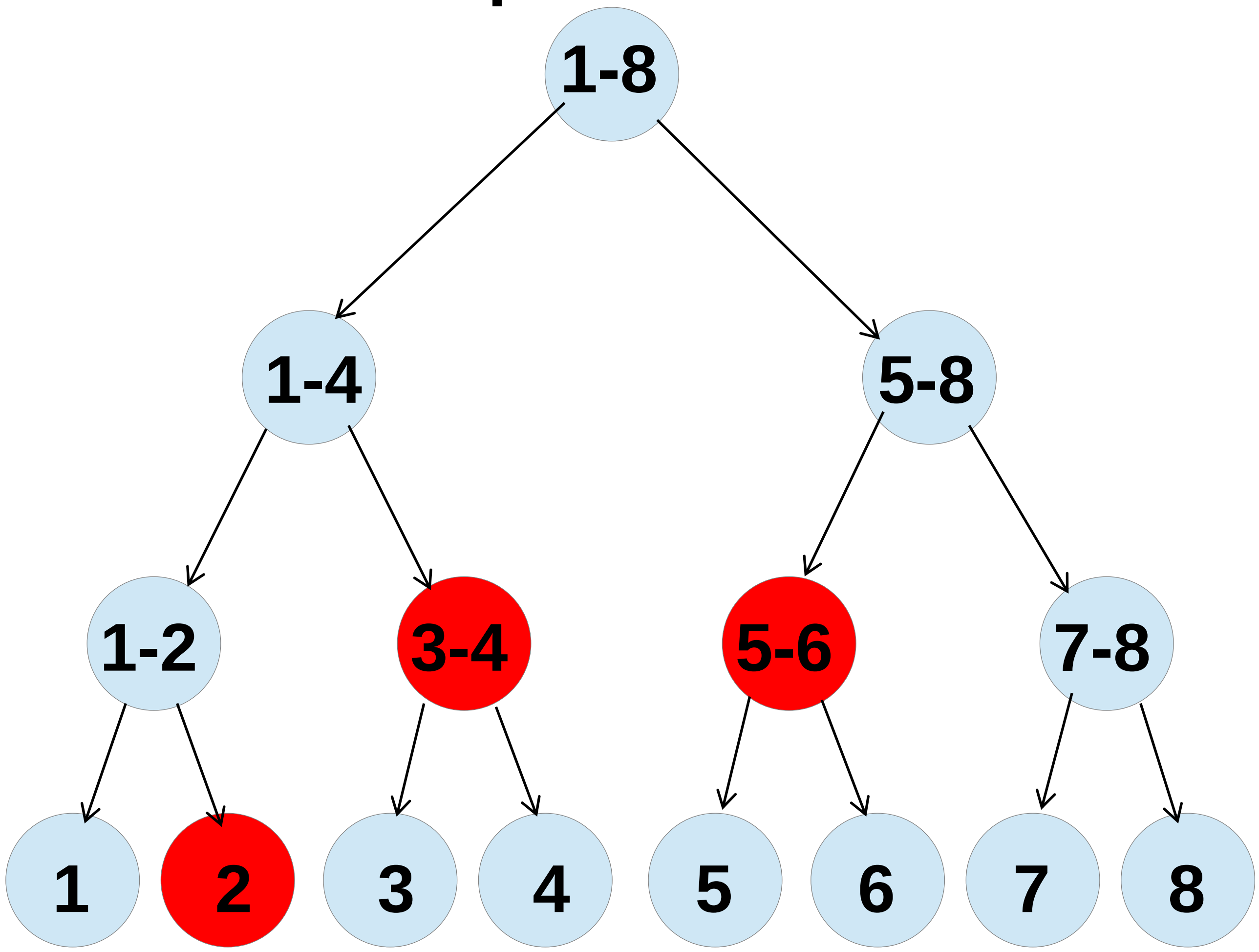
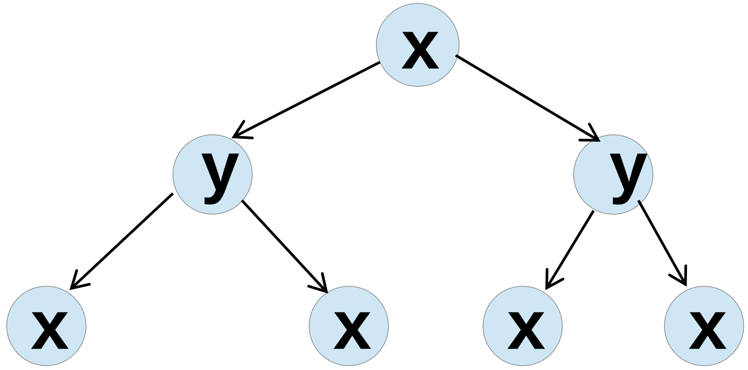


K-d tree

- Space-partitioning data structure for organizing points in k-dimensional space
- Used to store a set of points
- Partitioning alternating between the dimensions



A 2-dimensional k-d tree

K-d tree
Operations

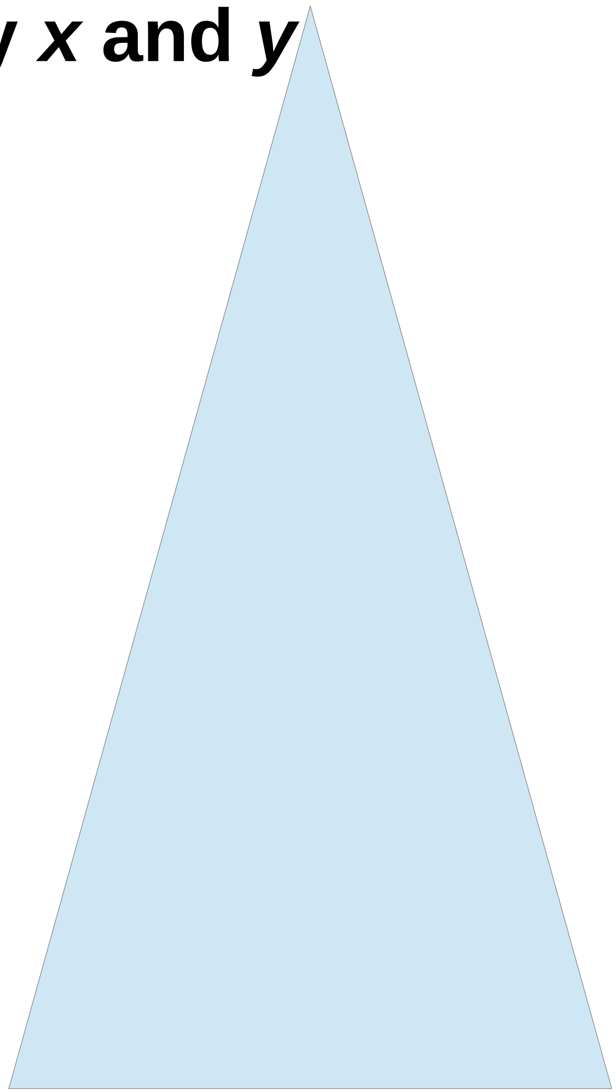
- Search – O($n^{1/d}$), where d is the number of
- dimensions
- Preprocessing time – O(n log n)
- Static!

K-d tree: an important fact

- We can represent any interval using square root of *n* nodes of the tree.

Data Structure

2-dimensional
k-d tree
by *x* and *y*



Data Structure

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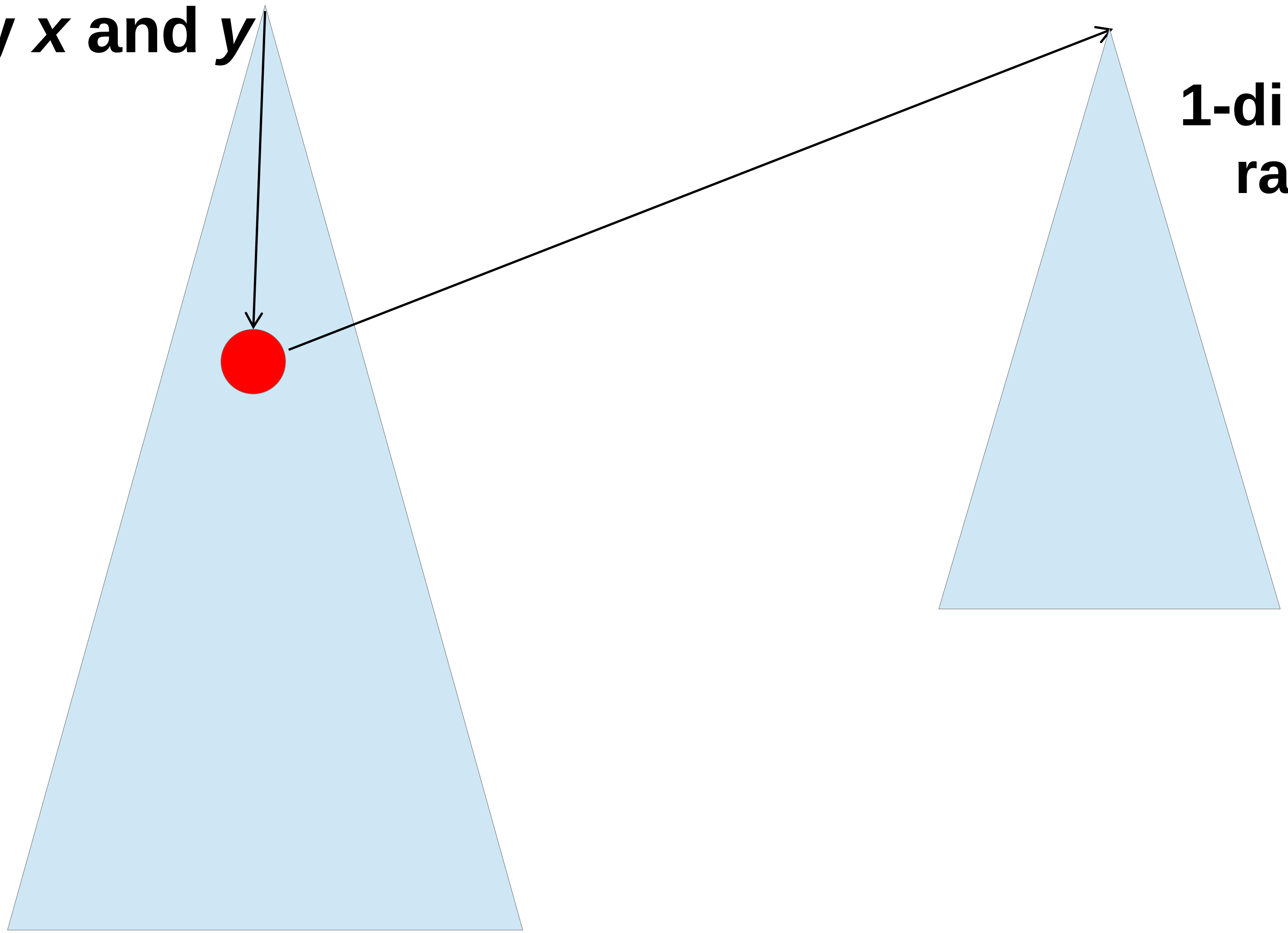
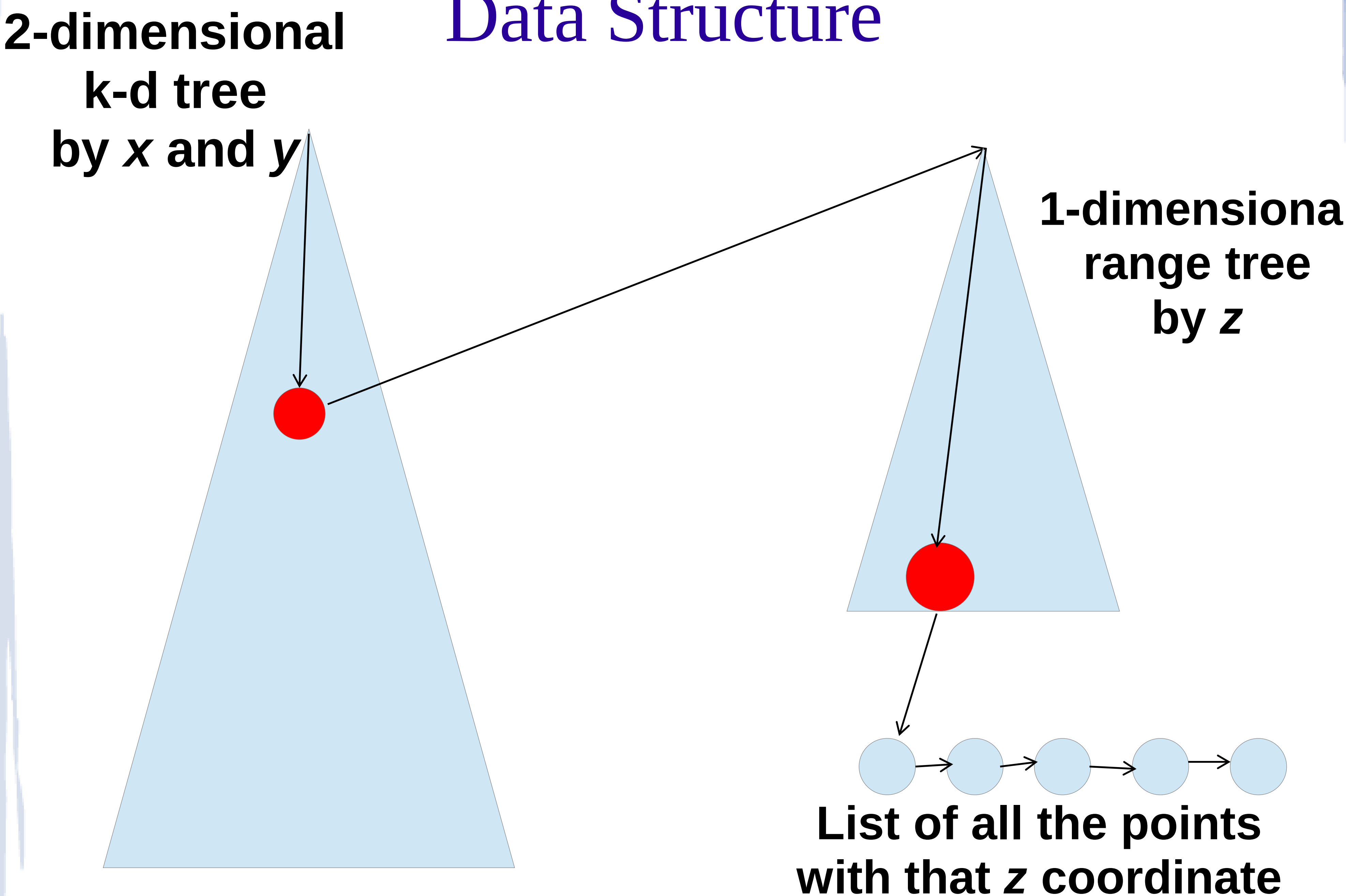


1-dimensional range tree by *z*

Data Structure

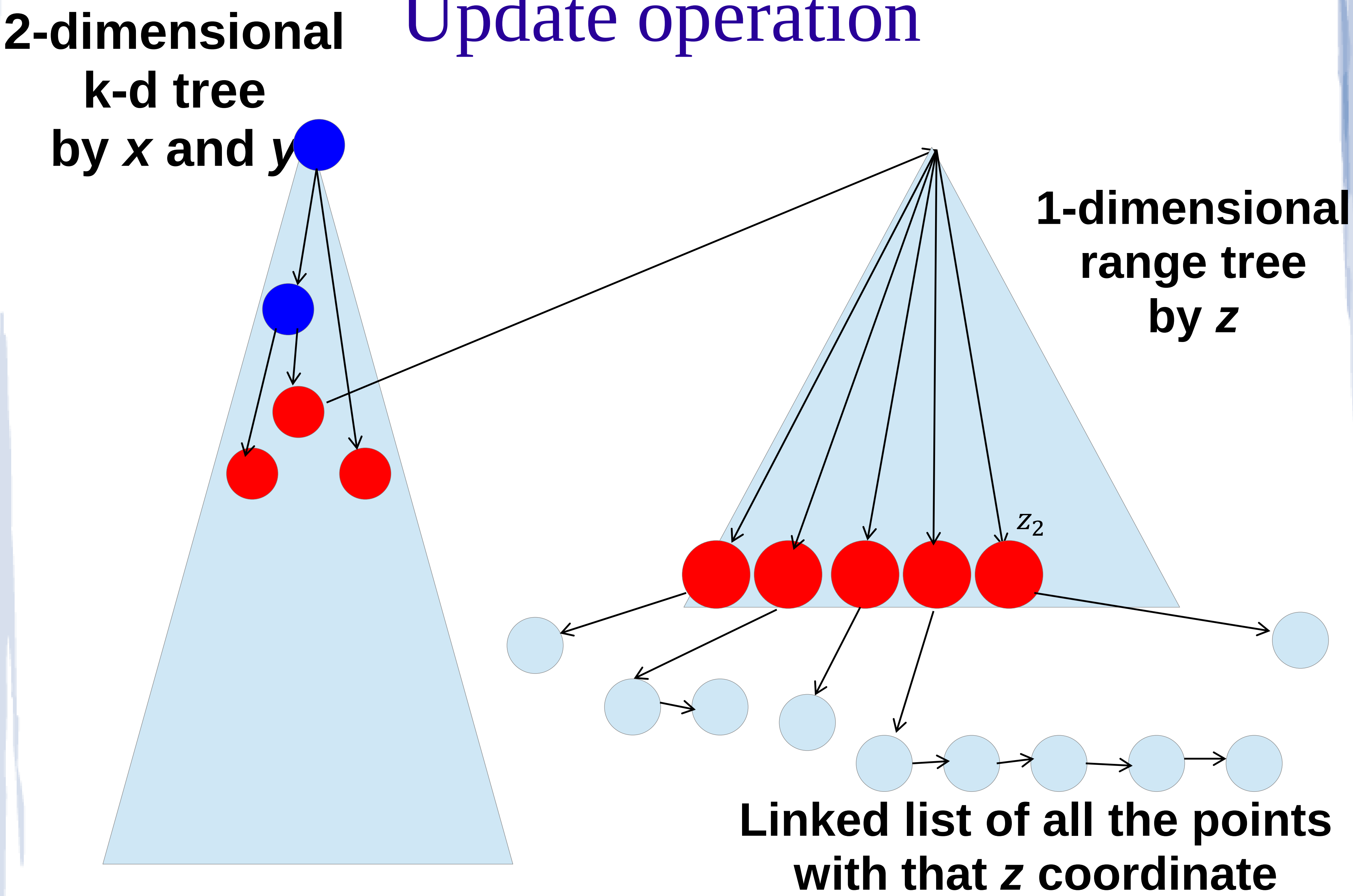


Update operation

Update: For all points in the interval $[x_{1,}x_2];[y_{1,}\infty];[-\infty,z_2]$ increase their *z* coordinate to z_2.

Step 1: Divide the interval $[x_{1,}x_2];[y_{1,}\infty]$ in $n^{1/2}$ nodes of the k-d tree. For each such node and each of their ancestors, handle the range tree that corresponds to them by moving the contents of each leaf one by one.

Update operation



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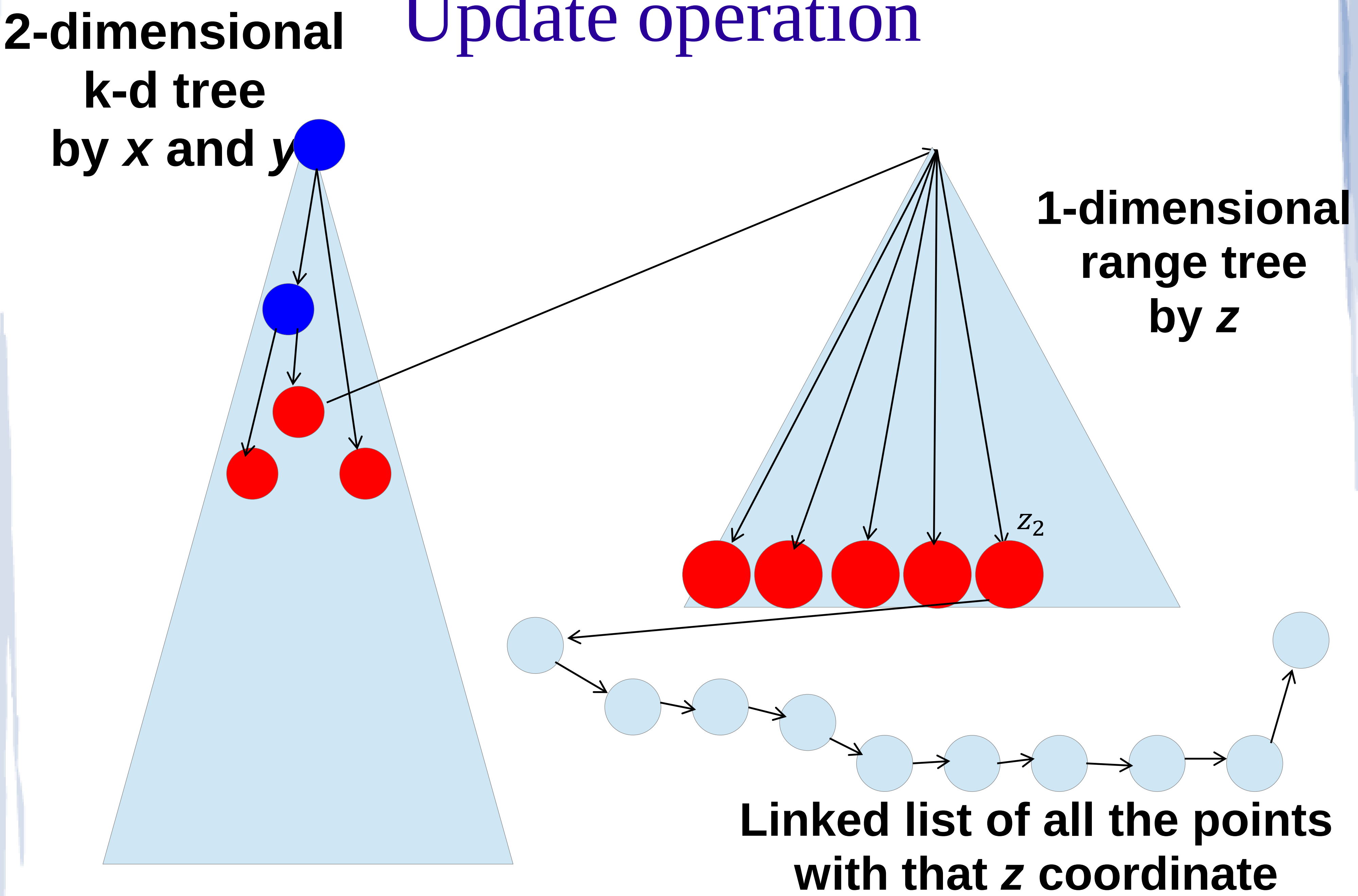

Update operation

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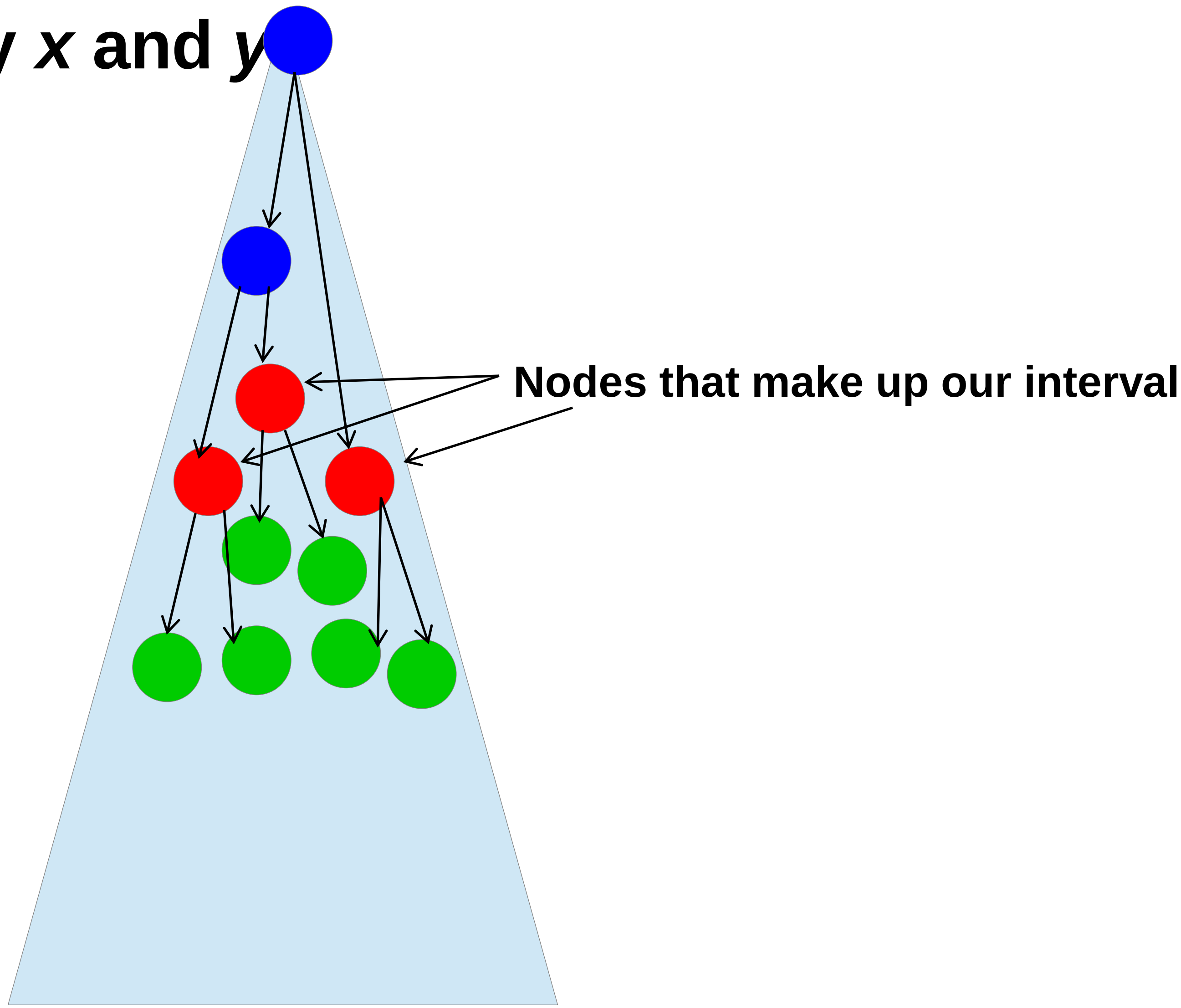
Step 2: For each of the children of these nodes, update the To Do label.

Update operation

2-dimensional
k-d tree
by *x* and *y*



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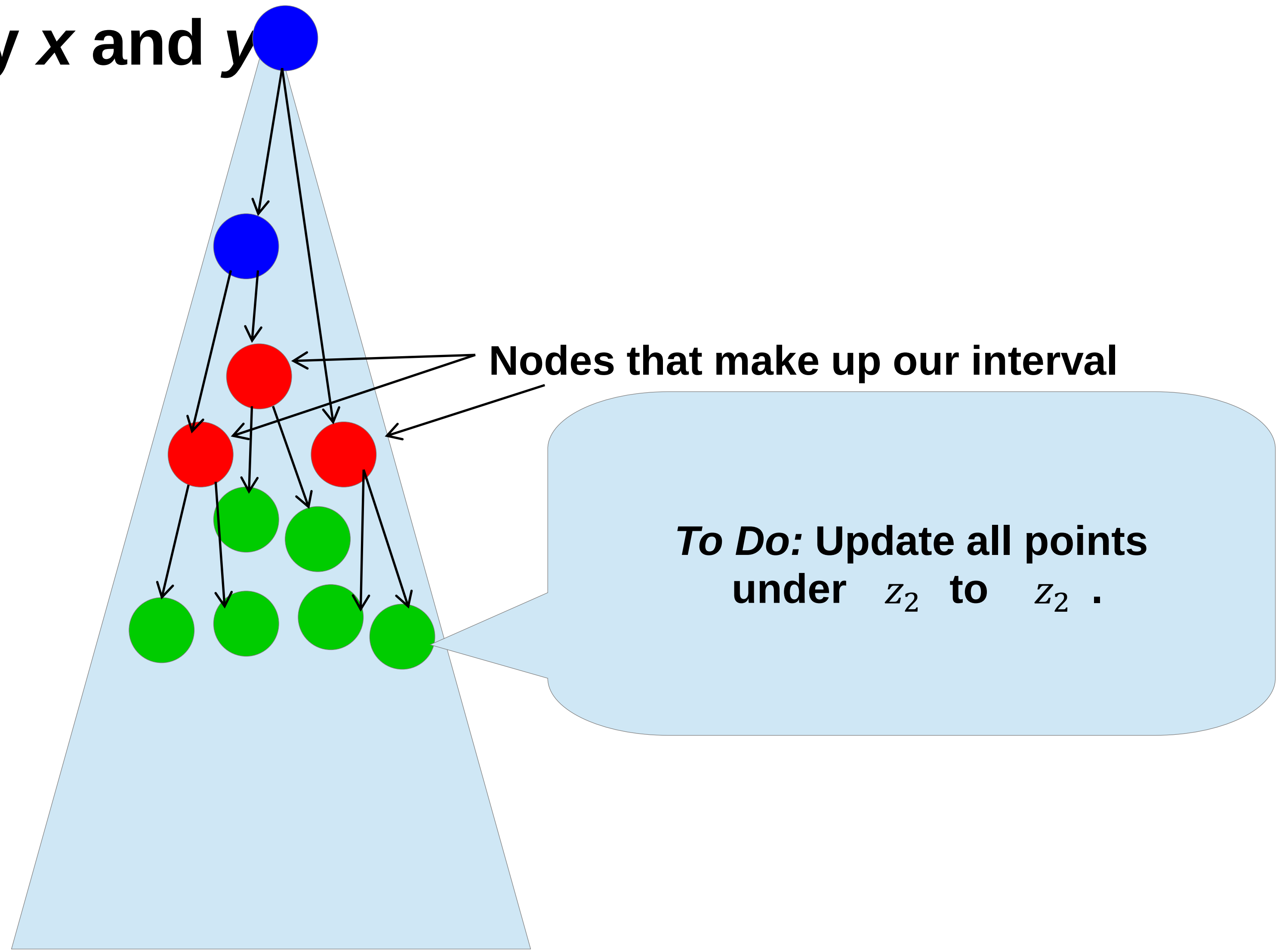

Problem

K-d trees are static, and we might need to insert new points!

Solution – the binary static to dynamic transformation

- Due to Saxe and Benteley, 1979
- O(log n) overhead for each operation

The binary static to dynamic transformation – how it works

- Suppose the static data structure is of type A.
- Keep at most *log n* data structures of type A, dividing the points in them in powers of 2.
- 19 points →
- 16 points in A1, 2 points in A2, 1 point in A3.

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The binary static to dynamic transformation
Insertion of a new point

- Suppose we insert a new point now:
- Destroy some of the data structures as necessary to create a new one with all of their points plus the new one.

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Notation
The Point Projection Problem

Total number of points	N
Number of update operations	M
Number of points reported in a query	K

Results
The Point Projection Problem

	2D case	3D case
Query	$O(\log N \log(N+M)+K)$	$O(\sqrt{N}\log_{(} N+M)+K)$
Insertion	$O(\log^2 N \log(N+M))$	$O(\sqrt{N}\log^2 N \log(N+M))$
Update	$O(\log^2 N \log(N+M))$	$O(\sqrt{N}\log^2 N \log(N+M))$
Storage	$O(N \log N \log(N+M))$	$O(N \log N \log(N+M))$
Preprocessing time	$O(N \log^2 N \log(N+M))$	$O(N\sqrt{N}\log^2 N \log(N+M))$

Notation
The Segment Intersection Problem

Number of segments	n
Number of points reported in a query	k

Results
The Segment Intersection Problem

	Static	Dynamic Trivial solution 1	Dynamic Trivial solution 2	Dynamic Our solution
Query	$O((k+1)\log n \log^* n)$	$O(\log^2 n + k)$	$\Omega(n \log n)$	$O(\sqrt{n}\log n + k \log^3 n)$
Insertion	Not supported	$\Omega(n \log^3 n)$	$O(1)$	$O(\sqrt{n}\log^3 n)$
Storage	$O(n \log n)$	$O(n \log^2 n)$	$O(n \log n)$	$O(n \log^2 n)$
Pre-processing time	$O(n \log n)$	$O(n \log^2 n)$	$O(n)$	$O(n\sqrt{n}\log^3 n)$

Conclusions

- Goal: designing a data structure that supports a set of axis-aligned segments and the following operations on them:
  - Insert a new segment
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- Goal: designing a data structure that supports a set of axis-aligned segments and the following operations on them:
  - Insert a new segment
  - Report all intersection points of pairs of segments inside a query rectangle
- Results:
  - sublinear time complexity of both operations, subquadratic storage space and subquadratic preprocessing time

Future work

- Getting rid of the square root of n factor

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- Getting rid of the square root of n factor
- Handling axis-aligned rectangles

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- Getting rid of the square root of n factor
- Handling axis-aligned rectangles
- Handling deletions of segments

Future work
Long-term goals

- Handling segments with arbitrary orientation

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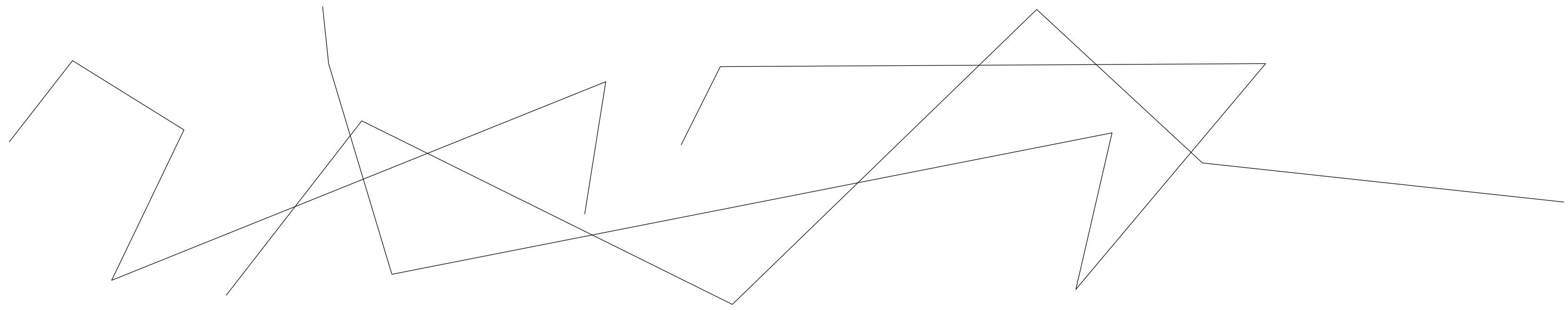
- Handling segments with arbitrary orientation
- Handling sequences of segments



Future work
Long-term goals

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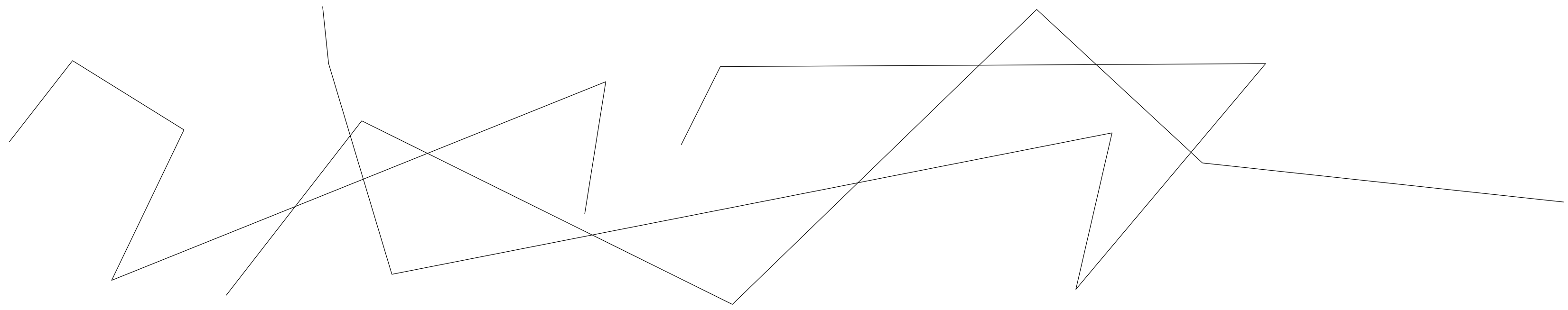
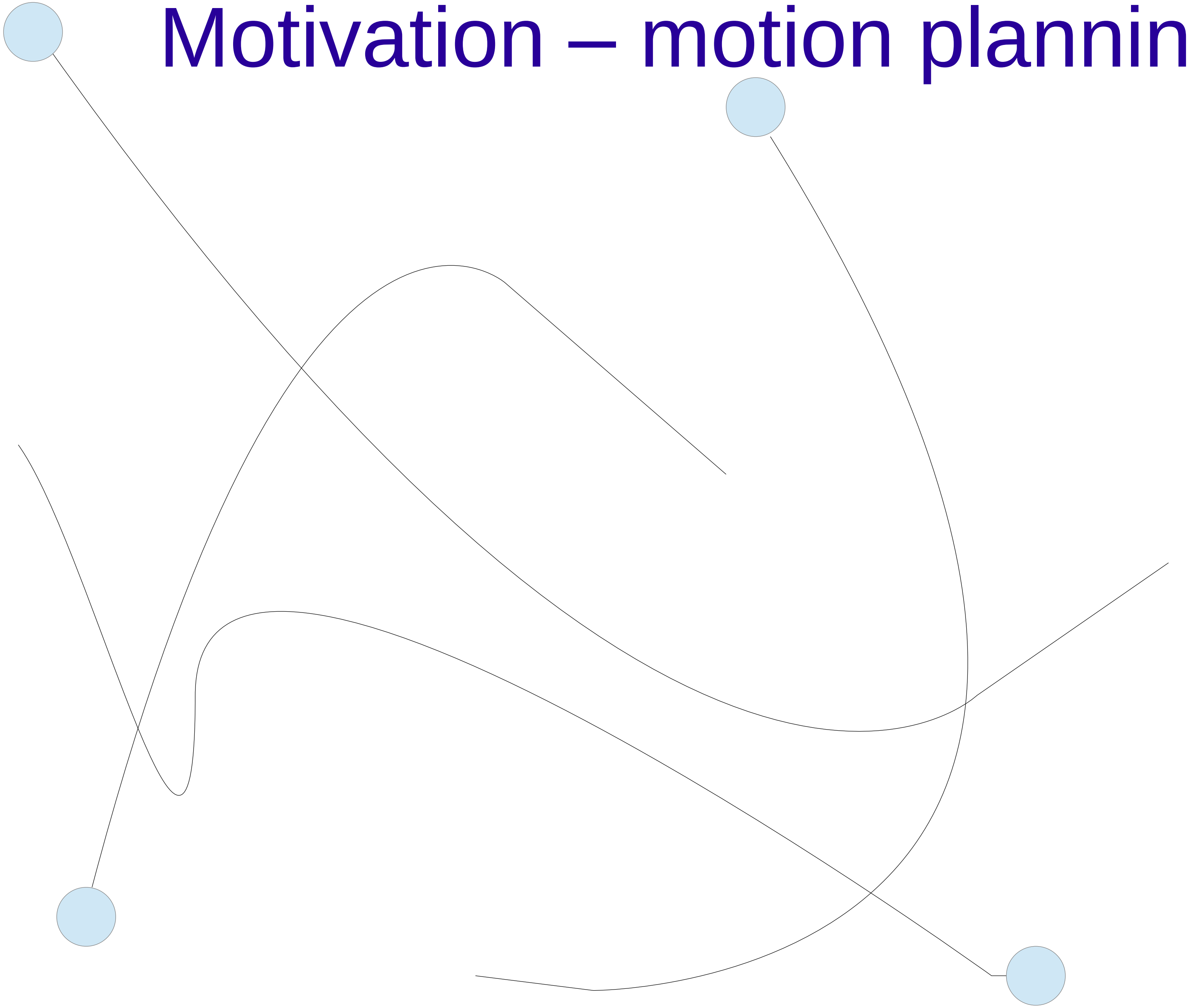
- Handling sequences of segments



- Handling intersections of curves

Thank you for your attention!

Motivation – motion planning



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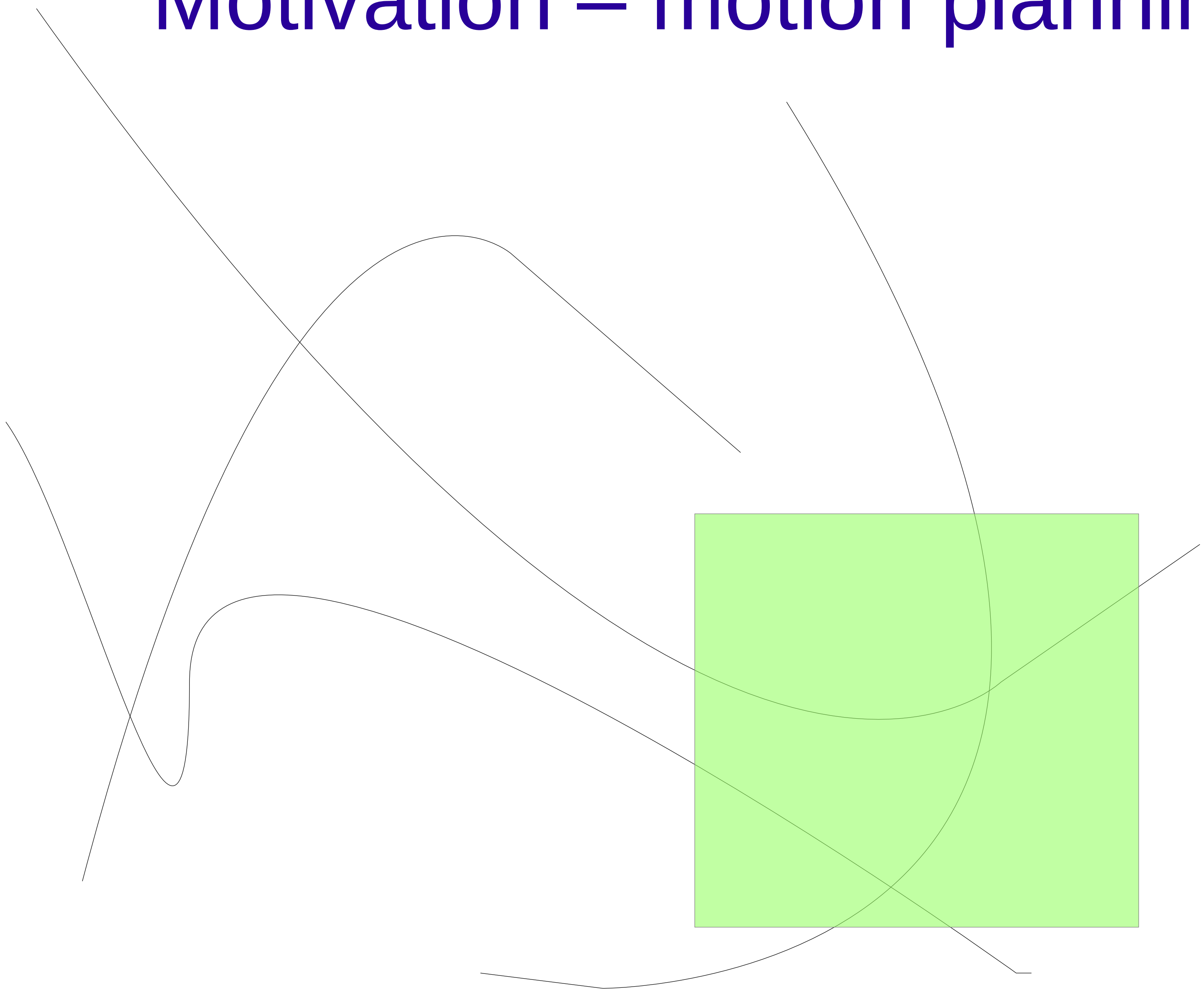

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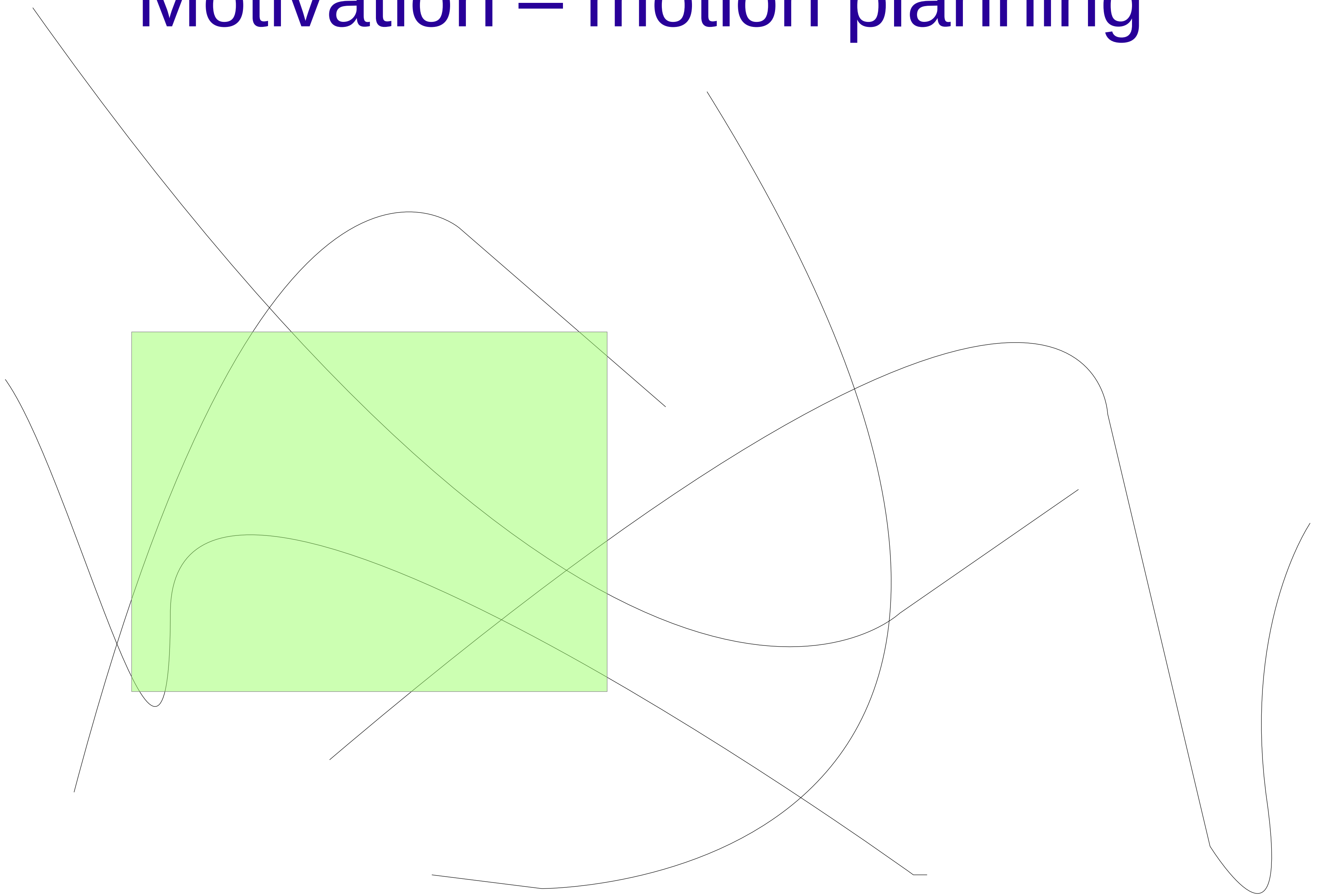

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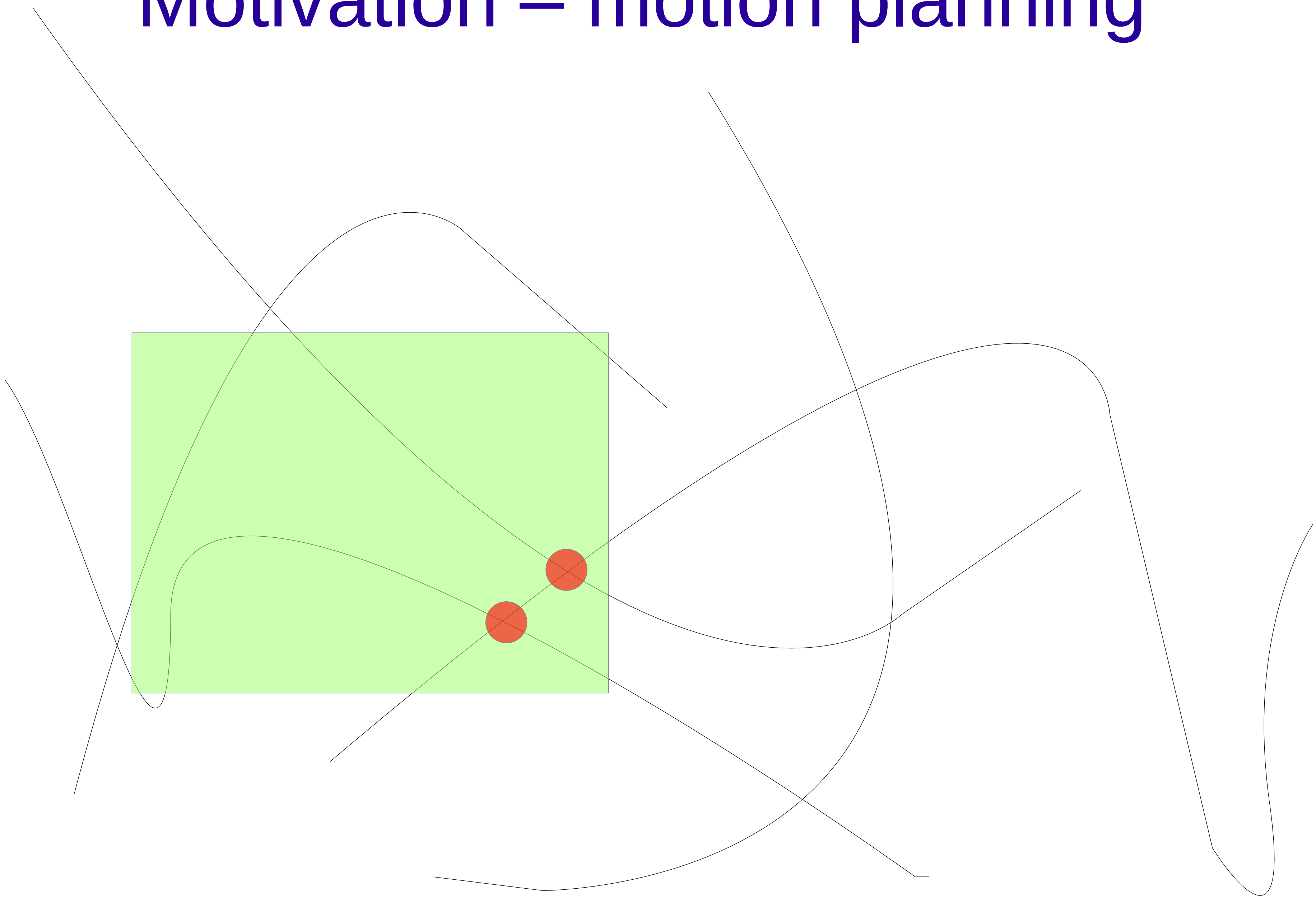
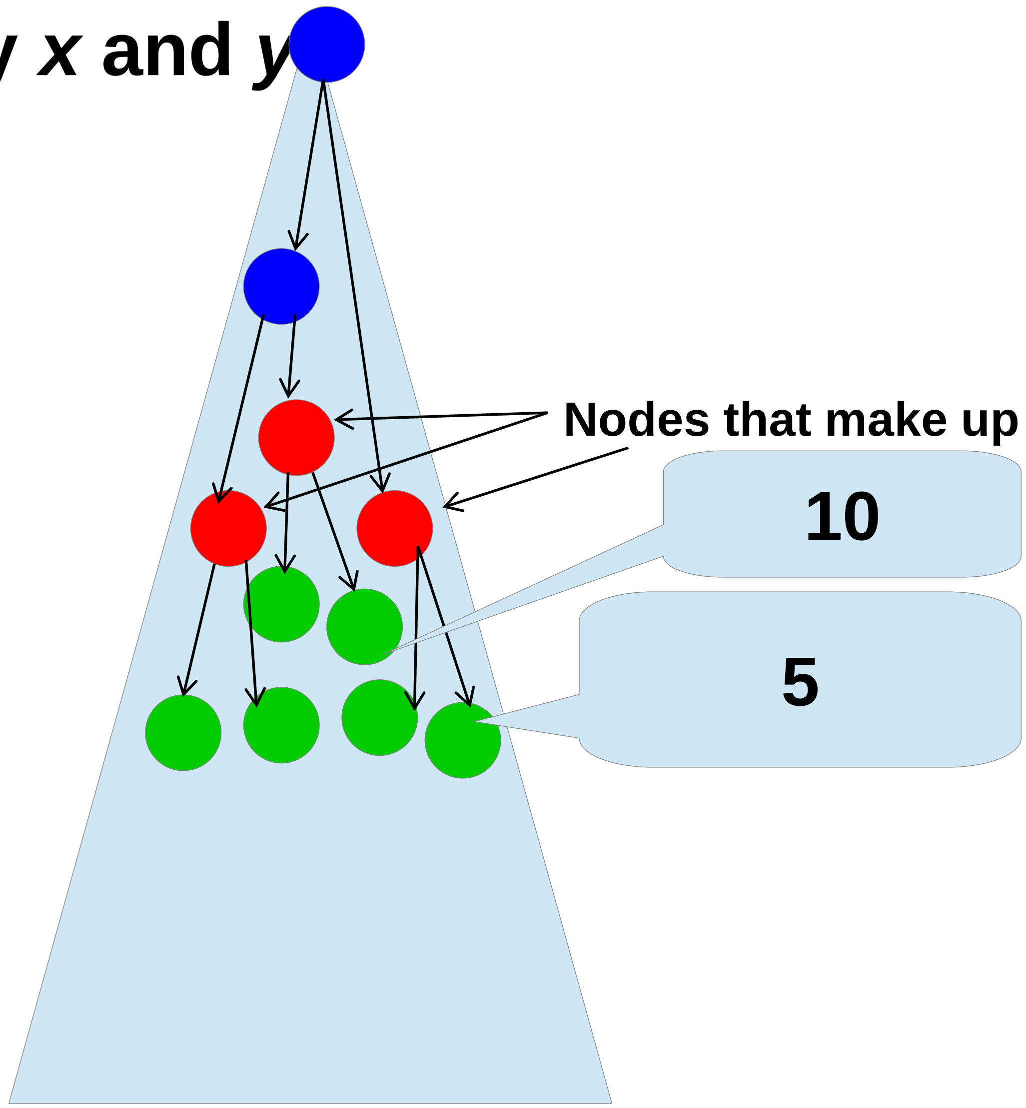


Update operation

2-dimensional k-d tree by *x* and *y*



Suppose z_2 = 8

Update operation

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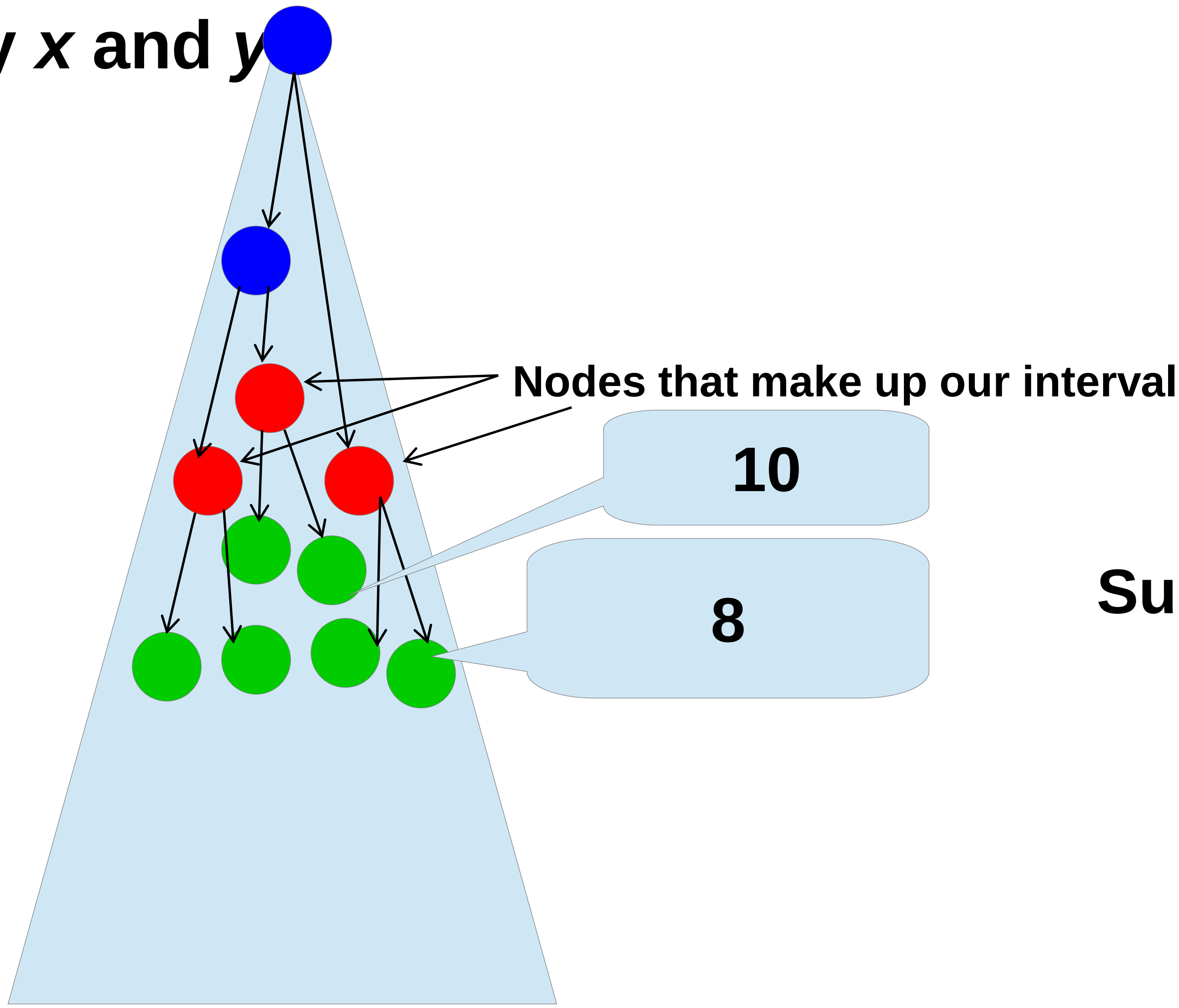
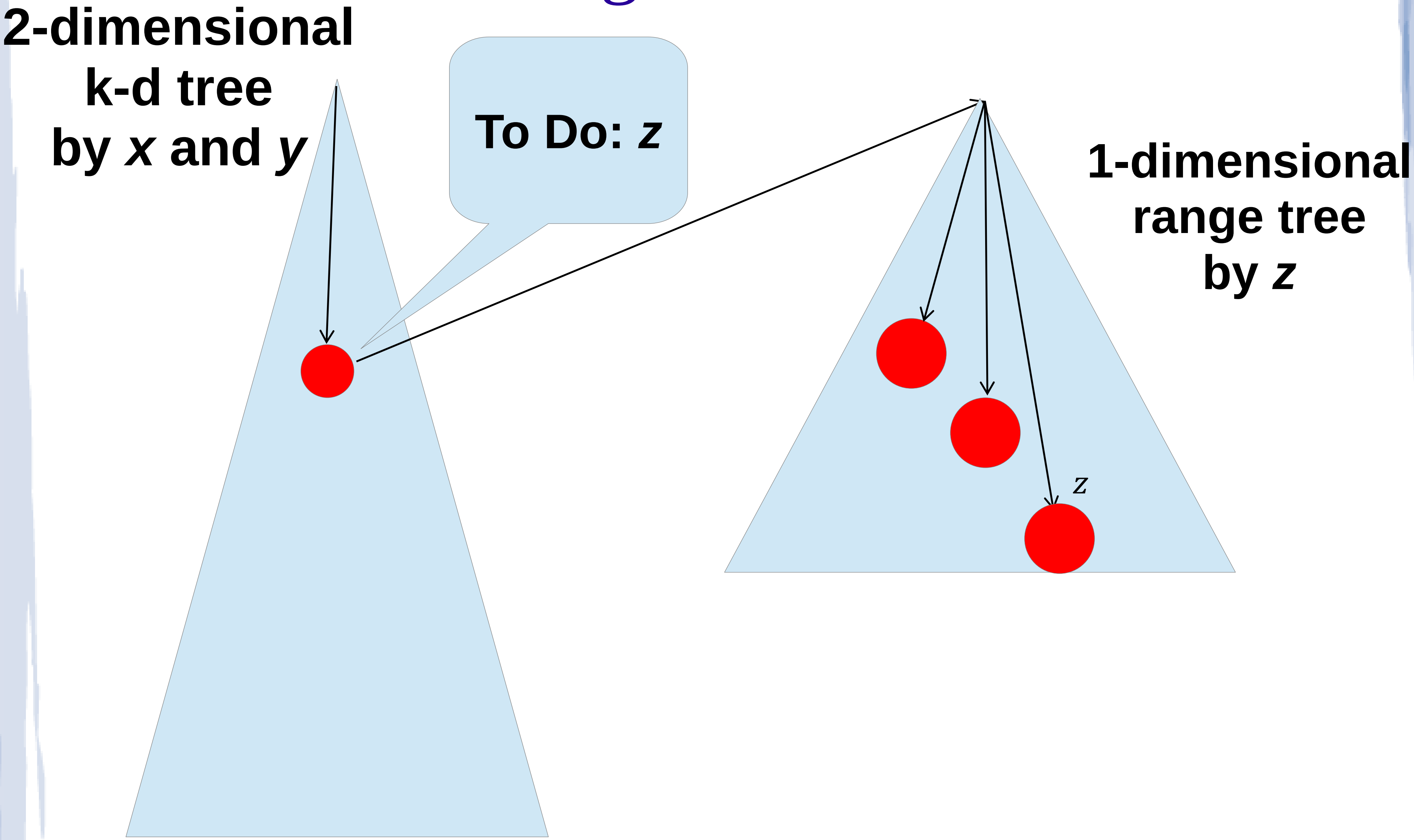


Suppose z_2 **= 8**

Handling *To Do* labels



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