Nonexistence of (9, 112, 4) and (10, 224, 5) binary orthogonal arrays

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Orthogonal arrays

For any two points x and y of the binary Hamming space $H(n, 2)$ by $d(x, y)$ we denote the Hamming distance and by $< x, y >= 1 - \frac{2d(x, y)}{n}$ their inner product.

Definition 1. A $M \times n$ matrix C with elements from $H(n, 2)$ is orthogonal array with strength τ $(0 \leq \tau \leq n)$, if every $M \times \tau$ submatrix of C contains each ordered τ-tuple of $H(n, 2)$ exactly $\lambda = M/2^\tau$ times. he value λ is called index of C.

We denote the orthogonal array C by (n, M, τ).

τ - strength, n - length, M - cardinality.

Distance distribution of C with respect to a point

Definition. Let $C \subset H(n,2)$ be a (n,M,τ) BOA and $y \in H(n,2)$ be fixed. The $(n+1)$-tuple $(w_0(y), w_1(y), \ldots, w_n(y))$, where

$$w_i(y) = |\{x \in C : d(y,x) = i \iff \langle x, y\rangle = t_{n-i} = 1 - \frac{2i}{n}\}|$$

for $i = 0, 1, \ldots, n$, is called distance distribution of a point y (distance distribution of C with respect to a point y).

Denote by
$\boldsymbol{P}(n,M,\tau)$ the set of distance distributions of any internal point $y \in C$,
$\boldsymbol{Q}(n,M,\tau)$ the set of distance distributions of any external point $y \in H(n,2), y \notin C$.
Denote also $\boldsymbol{W}(n,M,\tau) = P(n,M,\tau) \cup Q(n,M,\tau)$.

Computing distance distribution of a BOA

Theorem. Let $C \subset H(n,2)$ be an orthogonal array of parameters (n, M, τ) and $y \in H(n,2)$ be fixed. Then

- if $y \in C$, then the distance distribution $p(y)$ of C satisfies

$$\sum_{i=0}^{n} p_i(y) \left(1 - \frac{2i}{n}\right)^k = b_k|C|, k = 0, 1, \ldots, \tau,$$

- if $y \notin C$, the distance distribution $q(y)$ of C satisfies

$$\sum_{i=1}^{n} q_i(y) \left(1 - \frac{2i}{n}\right)^k = b_k|C|, k = 0, 1, \ldots, \tau,$$

where b_k is the first coefficient in the expansion of t^k in terms of the normalized Krawtchouk polynomials,
Krawtchouk polynomials are the zonal spherical functions of the $H(n,2)$.

Connections between (n, M, τ) and its derivates

Theorem. Let $\tau < n$ and C be a (n, M, τ) orthogonal array with distance distribution $W = (w_0, w_1, \ldots, w_n)$. Removing any column of C yields an orthogonal array C' of parameters $(n-1, M, \tau)$. Let $W' = (w'_0, w'_1, \ldots, w'_{n-1})$ be the distance distribution of C'.

$$\left| \begin{array}{l} x_i + y_i = w_i, i = 1, 2, \ldots, n-1 \\ x_{i+1} + y_i = w'_i, i = 0, 1, \ldots, n-1 \\ y_0 = w_0 \\ x_n = w_n \\ x_i, y_i \in \mathbb{Z}, x_i \geq 0, y_i \geq 0, i = 0, \ldots, n \end{array} \right. \tag{1}$$

with variable $\{x_i, y_i\}, i = 0, \ldots, n$. If the orthogonal array $C \subset H(n, 2)$ of parameters (n, M, τ) and distance distribution W exists then this system has a solution.

Connections between (n, M, τ) and its derivates

Theorem (continuation). Furthermore, let $(x_0^{(r)} = 0, x_1^{(r)}, \ldots, x_n^{(r)}; y_0^{(r)}, y_1^{(r)}, \ldots, y_{n-1}^{(r)}, y_n^{(r)} = 0)$, $r = 1, \ldots, s$, be all s solutions for the system for all possible C', obtained from C when removing an arbitrary column. If the orthogonal array $C \subset H(n, 2)$ of parameters (n, M, τ) and distance distribution W exists then the system

$$\left| \begin{array}{lllll} k_1 & +k_2 & +\ldots & +k_s & = n \\ k_1 x_1^{(1)} & +k_2 x_1^{(2)} & +\ldots & +k_s x_1^{(s)} & = w_1 \\ k_1 x_2^{(1)} & +k_2 x_2^{(2)} & +\ldots & +k_s x_2^{(s)} & = 2w_2 \\ \vdots & & & & \\ k_1 x_n^{(1)} & +k_2 x_n^{(2)} & +\ldots & +k_s x_n^{(s)} & = nw_n \\ k_j \in \mathbb{Z}, & k_j \geq 0, & j = 1, & \ldots, s & \end{array} \right. \qquad (2)$$

has a solution with respect to the unknowns $k_1, k_2, \ldots, k_s$.

Connections between (n, M, τ) and its derivates

The construction is as follows:

	$W' = (w'_0, w'_1, \ldots, w'_{n-1})$ $C' = (n-1, M, \tau)$
0 0 $\vdots$ 0	$Y = (y_0, y_1, \ldots, y_{n-1})$ $C_0 = (n-1, M/2, \tau-1)$
1 1 $\vdots$ 1	$X = (x_1, x_2, \ldots, x_n)$ $C_1 = (n-1, M/2, \tau-1)$

$W = (w_0, w_1, \ldots, w_n)$
$C = (n, M, \tau)$

	$C' = (n-1, M, \tau)$
1 1 $\vdots$ 1	$C_0 =$ $(n-1, M/2, \tau-1)$
0 0 $\vdots$ 0	$C_1 =$ $(n-1, M/2, \tau-1)$

$\widehat{W} = (\widehat{w}_0, \widehat{w}_1, \ldots, \widehat{w}_n)$
$C^{1,0} = (n, M, \tau)$

Connections between (n, M, τ) and its derivates

- **Condition.** $Y \in W(n-1, M/2, \tau-1)$.
- **Condition.** $\overline{Y} \in W(n-1, M/2, \tau-1)$.
- **Condition.** $X \in W(n-1, M/2, \tau-1)$.
- **Condition.** $\overline{X} \in W(n-1, M/2, \tau-1)$.
- **Condition.** If the distance distribution of C with respect to $c = \mathbf{0} \in H(n,2)$ is $W = (w_0, w_1, \ldots, w_{n-1}, w_n) = (y_0, x_1 + y_1, \ldots, x_{n-1} + y_{n-1}, x_n)$, the distance distribution of $C^{1,0}$ with respect to the same point is $\widehat{W} = (x_1, x_2 + y_0, \ldots, x_n + y_{n-2}, y_{n-1})$, i.e. $\widehat{W} \in W(n, M, \tau)$.

Further Connections between (n, M, τ) and its derivates

Let n, M and $3 \leq \tau < n+1$ be fixed.

After applying the main algorithm for every $W \in W(n, M, \tau)$ we know all remaining couples (W, W') and for every such couple we have an uniquely determined corresponding couple (Y, X).

We see that $X, Y \in W(n-1, M/2, \tau-1)$ but $W' \in W(n-1, M, \tau)$, i.e. W' has strength τ which is bigger than the strengths of X and Y.

The natural continuation is to remove a column in C' and investigate when the condition above is possible.

Further Connections between (n, M, τ) and its derivates

| $C' - (n-1, M, \tau)$, W'

| $C'' - (n-2, M, \tau)$, W''

0	0	$R = (r_0, r_1, \ldots, r_{n-2})$	C_0, C_0'
$\vdots$	$\vdots$	$A_0 - (n-2, M/4, \tau-2)$	Y, Y'
0	0		
0	1	$Z = (z_1, z_2, \ldots, z_{n-1})$	
$\vdots$	$\vdots$	$A_1 - (n-2, M/4, \tau-2)$	
0	1		
1	0	$U = (u_0, u_1, \ldots, u_{n-2})$	C_1, C_1'
$\vdots$	$\vdots$	$B_0 - (n-2, M/4, \tau-2)$	X, X'
1	0		
1	1	$V = (v_1, v_2, \ldots, v_{n-1})$	
$\vdots$	$\vdots$	$B_1 - (n-2, M/4, \tau-2)$	
1	1		

Fig.1

Further Connections between (n, M, τ) and its derivates

- For every $W \in W(n, M, \tau)$ we know the all remaining triples (W', Y, X) and for every such triple we have the sets $\{(Y, Y', R, Z)\}$ and $\{(X, X', U, V)\}$ of all possible distance distributions of the relatives BOAs which can be obtained from the considering BOA C with this distance distribution $W \in W(n, M, \tau)$.
- **Note** $R, \overline{R} \in W(n-2, M/4, \tau-2)$.
- **Note** $Z, \overline{Z} \in W(n-2, M/4, \tau-2)$.
- **Note** $U, \overline{U} \in W(n-2, M/4, \tau-2)$.
- **Note** $V, \overline{V} \in W(n-2, M/4, \tau-2)$.

Further Connections between (n, M, τ) and its derivates

- **Note** $C_0^{1,0}$ is a $(n-1, M/2, \tau-1)$ BOA. Its distance distributions with respect to $\mathbf{0}$ is $\widehat{Y} = (z_1, z_2 + r_0, \dots, z_{n-1} + r_{n-3}, r_{n-2})$. $\widehat{Y}, \overline{\widehat{Y}} \in W(n-1, M/2, \tau-1)$.
- **Note** $C_1^{1,0}$ is a $(n-1, M/2, \tau-1)$ BOA. Its distance distributions with respect to $\mathbf{0}$ is $\widehat{X} = (v_1, v_2 + u_0, \dots, v_{n-1} + u_{n-3}, u_{n-2})$. $\widehat{X}, \overline{\widehat{X}} \in W(n-1, M/2, \tau-1)$.
- **Note** The obtained BOA C'' of parameters $(n-2, M, \tau)$ has distance distribution $W'' = (w_0'', w_1'', \dots, w_{n-2}'') =$ $(r_0 + u_0 + z_1 + v_1,$ $r_1 + u_1 + z_2 + v_2, \dots, r_{n-2} + u_{n-2} + z_{n-1} + v_{n-1})$ $\in W(n-2, M, \tau)$.

Further Connections between (n, M, τ) and its derivates

- To obtain new relations we reorder the rows of C' (simultaneously reordering the rows of the whole C) as we first take the rows with first coordinate 0, then we put the rows with first coordinate 1, respectively and remove that first coordinate.
- The resulting C'' has the same distance distribution W'', but the derived BOAs with parameters $(n-1, M/2, \tau-1)$ are new.
- Let we denote them by D_0, D_1, D_0' and D_1' and let their distributions be G, H, G' and H', respectively.

Further Connections between (n, M, τ) and its derivates

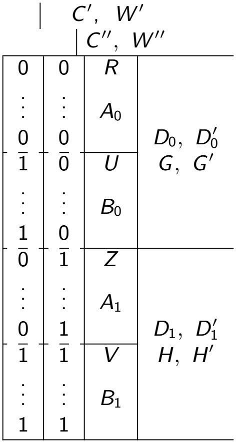

Fig.2

Further Connections between (n, M, τ) and its derivates

- **Theorem.** D_0 and D_1 are BOAs of parameters $(n-1, M/2, \tau-1)$ and distance distributions $G = (g_0, g_1, \ldots, g_{n-1}) = (r_0, r_1 + u_0, \ldots, r_{n-2} + u_{n-3}, u_{n-2})$ and $H = (h_1, h_2, \ldots, h_n) = (z_1, z_2 + v_1, \ldots, z_{n-1} + v_{n-2}, v_{n-1})$, i.e. $G, H \in W(n-1, M/2, \tau-1)$.

- **Condition.** $G, \overline{G}, \widehat{G}$ and $\overline{\widehat{G}} \in W(n-1, M/2, \tau-1)$.

- **Condition.** $H, \overline{H}, \widehat{H}$ and $\overline{\widehat{H}} \in W(n-1, M/2, \tau-1)$.

Further Connections between (n, M, τ) and its derivates

- **Theorem.** D'_0 and D'_1 are BOAs of parameters $(n-2, M/2, \tau-1)$ and distance distributions with respect to $c'' = \mathbf{0}'' \in H(n-2,2)$ are
  $G' = (g'_0, g'_1, \ldots, g'_{n-2}) = (r_0 + u_0, r_1 + u_1, \ldots, r_{n-2} + u_{n-2})$
  and
  $H' = (h'_1, h'_2, \ldots, h'_{n-1}) = (z_1 + v_1, z_2 + v_2, \ldots, z_{n-1} + v_{n-1})$,
  respectively, i.e. G', $H' \in W(n-2, M/2, \tau-1)$.

- **Condition.** $G', \overline{G'}, \widehat{G'}$ and $\overline{\widehat{G'}} \in W(n-2, M/2, \tau-1)$.

- **Condition.** $H', \overline{H'}, \widehat{H'}$ and $\overline{\widehat{H'}} \in W(n-2, M/2, \tau-1)$.

Further Connections between (n, M, τ) and its derivates

- Removing the second column of C to obtain a BOA C_2' with parameters $(n-1, M, \tau)$. Let $\tilde{W}'$ be the distance distribution of C_2' with respect to c'.

- **Theorem.** $\tilde{W}' = (r_0 + z_1, u_0 + r_1 + v_1 + z_2, \ldots, u_{n-3} + r_{n-2} + v_{n-2} + z_{n-1}, u_{n-2} + v_{n-1}) \in W(n-1, M, \tau)$.
- **Theorem.** The distance distribution of $(C_2')^{1,0}$ with respect to c' is $\widehat{\tilde{W}'} = (u_0 + v_1, r_0 + u_1 + z_1 + v_2, \ldots, r_{n-3} + u_{n-2} + z_{n-2} + v_{n-1}, r_{n-2} + z_{n-1}) \in W(n-1, M, \tau)$.

- **Condition.** $\tilde{W}', \overline{\tilde{W}'}, \widehat{\tilde{W}'}, \overline{\widehat{\tilde{W}'}}$ $\in$W(n-1,M,τ).

Further Connections between (n, M, τ) and its derivates

- We consider the effect of the permutation $(0 \to 1, 1 \to 0)$ in the first two columns (simultaneously). Denote the new BOA with $\tilde{C}$.

- **Theorem.** The distance distribution of $\tilde{C}$ with respect to c is $\tilde{W} = (v_1, u_0 + z_1 + v_2, r_0 + u_1 + z_2 + v_3, \ldots, r_{n-4} + u_{n-3} + z_{n-2} + v_{n-1}, r_{n-3} + u_{n-2} + z_{n-1}, r_{n-2}) \in W(n, M, \tau)$.

- **Condition.** $\tilde{W}$, $\overline{\tilde{W}}$, $\widehat{\tilde{W}}$ and $\overline{\widehat{\tilde{W}}}$ $\in$W(n,M,τ).

Further Connections between (n, M, τ) and its derivates

After all above checks, for every survival $W \in W(n, M, \tau)$ we have attached triples (W', Y, X)-(Y, Y', R, Z)-(X, X', U, V).
We now free the cut of the second column and thus consider all possible $n-1$ cuts of columns of C'. These cuts produce all possible pairs $\{(Y, Y', R, Z)\}$ - $\{(X, X', U, V)\}$. Let

$$(z_0^{(i)} = 0, z_1^{(i)}, \ldots, z_{n-2}^{(i)}, z_{n-1}^{(i)}; r_0^{(i)}, r_1^{(i)}, \ldots, r_{n-2}^{(i)}, r_{n-1}^{(i)} = 0), \;\; i = 1, \ldots, s,$$

$$(v_0^{(j)} = 0, v_1^{(j)}, \ldots, v_{n-2}^{(j)}, v_{n-1}^{(j)}; u_0^{(j)}, u_1^{(j)}, \ldots, u_{n-2}^{(j)}, v_{n-1}^{(j)} = 0), \;\; j = 1, \ldots, t,$$

are all solutions of system (1) for (Y, Y') and (X, X'). If (Z^i, R^i, V^j, U^j) satisfy all the conditions above then we denote them with $Z^{i,j}, R^{i,j}, V^{i,j}, U^{i,j}$.

Further Connections between (n, M, τ) and its derivates

Theorem. The nonnegative integers $k_{i,j}$, $i = 1, \ldots, s; j = 1, \ldots, t$, satisfy the following system of linear equations

$$
\left|
\begin{array}{ll}
\sum_{i,j} k_{i,j} & = n \\
\sum_{i,j} k_{i,j} r_0^{(i,j)} & = y_1 \\
\sum_{i,j} k_{i,j} r_1^{(i,j)} & = 2y_2 \\
\vdots & \\
\sum_{i,j} k_{i,j} r_{n-2}^{(i,j)} & = ny_{n-1} \\
\sum_{i,j} k_{i,j} u_0^{(i,j)} & = x_2 \\
\sum_{i,j} k_{i,j} u_1^{(i,j)} & = 2x_3 \\
\vdots & \\
\sum_{i,j} k_{i,j} u_{n-2}^{(i,j)} & = nx_n \\
k_{i,j} \in, k_{i,j} \geq 0, & i = 1, \ldots, s; j = 1, \ldots, t
\end{array}
\right.
$$

Results of the algorithm

Let $C = (n, M, \tau)$ be a BOA of targeted parameters, where $\tau \geq 3$.

- For every $W \in W(n, M, \tau)$ we have the sets of all feasible triples (W', Y, X). For every such triple we find the corresponding sets $\{(Y, Y', R, Z)\}$ and $\{(X, X', U, V)\}$ - part one of the algorithm.
- For every fixed W-(W', Y, X)-(Y, Y', R, Z)-(X, X', U, V) we check the required conditions - part 2.
- On every step we try to reduce the sets $P(n, M, \tau)$, $Q(n, M, \tau)$ and $W(n, M, \tau)$. The algorithm stops when no new rulings out are possible.
- If one of the set becomes empty this means nonexistence of the corresponding BOA.

Results of the algorithm

- For $(8, 56, 3)$ BOA the row with the distance distributions for $(3, 56, 3), (4, 56, 3), (5, 56, 3), (6, 56, 3), (7, 56, 3), (7, 56, 3)$ is:

  1, 8, 19, 54, 110 (112), 248 (264).

- For $(9, 112, 4)$ BOA the row with the distance distributions for $(4, 112, 4), (5, 112, 4), (6, 112, 4), (7, 112, 4), (8, 112, 4), (9, 112, 4)$ is

  1, 8, 16, 18, 34, 0 (33).

- **Theorem.** There exist no binary orthogonal arrays of parameters $(9, 112, 4)$ and $(10, 224, 5)$.

Results

Minimum possible index λ of binary orthogonal array of length n, $7 \leq n \leq 13$, and strength τ, $4 \leq \tau \leq 10$ up to MS results (2016)

n / τ	4	5	6	7	8	9	10
7	${}^{sz}4$	2	1	1			
8	4^{c}	${}^{sz}4$	2	1	1		
9	8^{ms}	4^{c}	4	2	1	1	
10	8^{bms}	8^{ms}	8^{kh}	4	2	1	1
11	8^{bms}	8^{bms}	8^{c}	8^{kh}	4	2	1
12	8^{bkms}	8^{bms}	12-16	8^{c}	8^{kh}	4	2
13	8	8^{bkms}	16	12-16	16^{kh}	8^{kh}	4

Thank you for your attention!

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