Mac Williams identities for linear codes as Riemann-Roch conditions

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The genus of a linear code

Let C be an $\mathbb{F}_q$-linear $[n, k, d]$-code.

The genus of C is the deviation $g := n + 1 - k - d$ from the equality in the Singleton bound $n + 1 - k - d \geq 0$.

Let us denote by $g^\perp = k + 1 - d^\perp$ the genus of the dual code $C^\perp = \left\{ a \in \mathbb{F}_q^n \mid \langle a, c \rangle = \sum_{i=1}^{n} a_i c_i = 0 \text{ for } \forall c \in C \right\}$.

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The homogeneous weight enumerator of a linear code

If $\mathcal{W}_C^{(w)}$ is the number of the words $c \in C$ of weight $1 \le w \le n$ then $\mathcal{W}_C(x,y) = x^n + \sum_{w=d}^{n} \mathcal{W}_C^{(w)} x^{n-w} y^w$ is called the homogeneous weight enumerator of C.

Denote by $\mathcal{M}_{n,s}(x,y) = x^n + \sum_{w=s}^{n} \mathcal{M}_{n,s}^{(w)} x^{n-w} y^w$ with $\mathcal{M}_{n,s}^{(w)} = \binom{n}{w} \sum_{i=0}^{w-s} (-1)^i \binom{w}{i} (q^{w+1-s-i} - 1)$ the homogeneous weight enumerator of an MDS-code with parameters $[n, n+1-s, s]$.

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The ζ-polynomial and the ζ-function of a linear code

Theorem (Duursma - 1999): For any linear code C of genus $g \geq 0$ with dual $C^{\perp}$ of genus $g^{\perp} \geq 0$ there is a unique ζ-polynomial $P_C(t) = \sum_{i=0}^{g+g^{\perp}} a_i t^i \in \mathbb{Q}[t]$ with

$$\mathcal{W}_C(x, y) = \sum_{i=0}^{g+g^{\perp}} a_i \mathcal{M}_{n,d+i}(x, y) \text{ and } P_C(1) = 1.$$

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Let $X/\mathbb{F}_q \subset \mathbb{P}^N(\overline{\mathbb{F}_q})$ be a smooth irreducible curve of genus g, $P_1, \ldots, P_n \in X(\mathbb{F}_q) = X \cap \mathbb{P}^N(\mathbb{F}_q)$, $D = P_1 + \ldots + P_n$ and $G_1, \ldots, G_h$ be a complete set of representatives of the linear equivalence classes of the divisors of $\mathbb{F}_q(X)$ of degree $2g - 2 < m < n$ with $\mathrm{Supp}(G_i) \cap \mathrm{Supp}(D) = \emptyset$ for $\forall 1 \leq i \leq h$.

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The evaluation maps $\mathcal{E}_D : H^0(X, \mathcal{O}_X([G_i])) \to \mathbb{F}_q^n$, $\mathcal{E}_D(f) = (f(P_1), \ldots, f(P_n))$ of the global sections f of the line bundles on X, associated with G_i are $\mathbb{F}_q$-linear.

Their images $C_i = \mathcal{E}_D H^0(X, \mathcal{O}_X([G_i])) \subset \mathbb{F}_q^n$ are $\mathbb{F}_q$-linear codes of genus $g_i \leq g$, known as algebro-geometric Goppa codes.

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If $|X(\mathbb{F}_{q^r})|$ is the number of the $\mathbb{F}_{q^r}$-rational points $X(\mathbb{F}_{q^r}) := X \cap \mathbb{P}^N(\mathbb{F}_{q^r})$ of X then the formal power series $\zeta_X(t) := \exp\left(\sum_{r=1}^{\infty} |X(\mathbb{F}_{q^r})| \frac{t^r}{r}\right)$ is called the ζ-function of X.

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The absolute Galois group $\mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$ acts on any smooth irreducible curve $X/\mathbb{F}_q \subset \mathbb{P}^N(\overline{\mathbb{F}_q})$ with finite orbits and $\deg \mathrm{Orb}_{\mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)}(x) := \left|\mathrm{Orb}_{\mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)}(x)\right|$.

The $\mathbb{Z}$-linear combinations $D = a_1\nu_1 + \ldots + a_s\nu_s$ of $\mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$-orbits $\nu_j \subset X$ are called divisors on X.

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Effective divisors of fixed degree

A divisor $D = a_1\nu_1 + \ldots + a_s\nu_s$ is effective if all of its non-zero coefficients $a_j > 0$ are positive.

There are finitely many $\mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$-orbits on X of fixed degree and, therefore, a finite number $\mathcal{A}_m(X) \in \mathbb{Z}^{\geq 0}$ of effective divisors on X of degree $m \in \mathbb{Z}^{\geq 0}$.

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Immediate consequences of the Riemann-Roch Theorem on a smooth irreducible curve $X/\mathbb{F}_q \subset \mathbb{P}^N(\overline{\mathbb{F}_q})$ of genus g are the Riemann-Roch Conditions

$$\mathcal{A}_m(X) = q^{m-g+1}\mathcal{A}_{2g-2-m}(X) + (q^{m-g+1} - 1)\mathrm{Res}_1(\zeta_X(t))$$

for $\forall m \geq g$ and the residuum $\mathrm{Res}_1(\zeta_X(t))$ of $\zeta_X(t)$ at $t = 1$.

Polarized Riemann-Roch Conditions

Definition: Formal power series $\zeta(t) = \sum_{m=0}^{\infty} \mathcal{A}_m t^m$ and $\zeta^{\perp}(t) = \sum_{i=0}^{\infty} \mathcal{A}^{\perp}_m t^m$ satisfy the Polarized Riemann-Roch Conditions $PRRC(g, g^{\perp})$ for some $g, g^{\perp} \in \mathbb{Z}^{\geq 0}$ if

$$\mathcal{A}_m = q^{m-g+1}\mathcal{A}^{\perp}_{g+g^{\perp}-2-m} + (q^{m-g+1} - 1)\mathrm{Res}_1(\zeta(t)) \quad \text{for} \quad \forall m \geq g,$$

$$\mathcal{A}_{g-1} = \mathcal{A}^{\perp}_{g^{\perp}-1} \quad \text{and}$$

$$\mathcal{A}^{\perp}_m = q^{m-g^{\perp}+1}\mathcal{A}_{g+g^{\perp}-2-m} + (q^{m-g^{\perp}+1} - 1)\mathrm{Res}_1(\zeta^{\perp}(t)) \quad \text{for } \forall m \geq g^{\perp},$$

where $\mathrm{Res}_1(\zeta(t))$, $\mathrm{Res}_1(\zeta^{\perp}(t))$ are the residuums at $t = 1$.

Riemann-Roch Conditions imply rationality

Note that $\mathrm{PRRC}(g, g^{\perp})$ imply the recurrence relations

$$\mathcal{A}_{m+2} - (q+1)\mathcal{A}_{m+1} + q\mathcal{A}_m = \mathcal{A}^{\perp}_{m+2} - (q+1)\mathcal{A}_{m+1} + q\mathcal{A}^{\perp}_m = 0$$

for $\forall m \geq g + g^{\perp} - 1$, which hold exactly when

$$\zeta(t) = \frac{P(t)}{(1-t)(1-qt)}, \quad \zeta^{\perp}(t) = \frac{P^{\perp}(t)}{(1-t)(1-qt)}$$

for polynomials $P(t), P^{\perp}(t) \in \mathbb{C}[t]$.

Theorem: Mac Williams identities for an $\mathbb{F}_q$-linear $[n, k, d]$-code C of genus $g := n + 1 - k - d \geq 0$ and its dual $C^{\perp} \subset \mathbb{F}_q^n$ of genus $g^{\perp} = k + 1 - d^{\perp} \geq 0$ are equivalent to the Polarized Riemann-Roch Conditions $\mathrm{PRRC}(g, g^{\perp})$ on their ζ-functions $\zeta_C(t)$, $\zeta_{C^{\perp}}(t)$.

Definition of Duursma's reduced polynomial

Proposition (KM - 2014): Let C be an $\mathbb{F}_q$-linear $[n, k, d]$-code of genus $g = n + 1 - k - d \geq 1$, whose dual $C^\perp$ is of genus $g^\perp = k + 1 - d^\perp \geq 1$. Then there is a unique Duursma's reduced polynomial $D_C(t) = \sum_{i=0}^{g+g^\perp-2} c_i t^i \in \mathbb{Q}[t]$, such that $\mathcal{W}_C(x, y) = \mathcal{M}_{n,n+1-k}(x, y) + \sum_{i=0}^{g+g^\perp-2} (q-1) c_i \binom{n}{d+i} (x-y)^{n-d-i} y^{d+i}$.

Corollary: The lower parts $\varphi_C(t) = \sum_{i=0}^{g-2} c_i t^i$, $\varphi_{C^\perp}(t) = \sum_{i=0}^{g^\perp-2} c_i^\perp t^i$ of Duursma's reduced polynomials $D_C(t) = \sum_{i=0}^{g+g^\perp-2} c_i t^i$, $D_{C^\perp}(t) = \sum_{i=0}^{g+g^\perp-2} c_i^\perp t^i$ and the number $c_{g-1} = c_{g^\perp-1}^\perp \in \mathbb{Q}$ determine uniquely

$$D_C(t) = \varphi_C(t) + c_{g-1}t^{g-1} + \varphi_{C^\perp}\left(\frac{1}{qt}\right) q^{g^\perp-1} t^{g+g^\perp-2},$$

$$D_{C^\perp}(t) = \varphi_{C^\perp}(t) + c_{g-1}t^{g^\perp-1} + \varphi_C\left(\frac{1}{qt}\right) q^{g-1} t^{g+g^\perp-2}.$$

Corollary: If C if an $\mathbb{F}_q$-linear code of genus $g \geq 1$ with Duursma's reduced polynomial $D_C(t) = \sum_{i=0}^{g+g^\perp-2} c_i t^i \in \mathbb{Q}[t]$, then

$$c_i \binom{n}{d+i} \in \mathbb{Z}^{\geq 0} \quad \text{for} \quad \forall 0 \leq i \leq g + g^\perp - 2.$$

A linear code $C \subset \mathbb{F}_q^n$ is non-degenerate if it is not contained in a coordinate hyperplane $V(x_i) = \{a \in \mathbb{F}_q^n \mid a_i = 0\}$ for some $1 \leq i \leq n$.

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An averaging interpretation of the coefficients of Duursma's reduced polynomial

Proposition: Let C be a non-degenerate $\mathbb{F}_q$-linear code with Duursma's reduced polynomial $D_C(t) = \sum_{i=0}^{g+g^{\perp}-2} c_i t^i \in \mathbb{Q}[t]$ and

$$\mathbb{P}(C)^{(\subseteq \beta)} = \{[a] \in \mathbb{P}(C) \subset \mathbb{P}(\mathbb{F}_q^n) \mid \mathrm{Supp}([a]) \subseteq \beta\}$$

for $\beta = \{\beta_1, \ldots, \beta_{d+i}\} \subset \{1, \ldots, n\}$ with $0 \leq i \leq g-1$. Then

$$c_i = \binom{n}{d+i}^{-1} \left(\sum_{\beta = \{\beta_1, \ldots, \beta_{d+i}\} \subset \{1, \ldots, n\}} \left| \mathbb{P}(C)^{(\subseteq \beta)} \right| \right)$$

is the average cardinality of an intersection of $\mathbb{P}(C)$ with $n-d-i$ coordinate hyperplanes in $\mathbb{P}(\mathbb{F}_q^n)$.

Probabilistic interpretations of the coefficients of Duursma's reduced polynomial

Proposition: Let C be an $\mathbb{F}_q$-linear code with Duursma's reduced polynomial $D_C(t) = \sum_{i=0}^{g+g^{\perp}-2} c_i t^i \in \mathbb{Q}[t]$. If $\pi^{(w)}_{\mathbb{P}(C)}$, respectively, $\pi^{(w)}_{\mathbb{P}(C^{\perp})}$ is the probability of $[b] \in \mathbb{P}(\mathbb{F}_q^n)$ with $\mathrm{wt}([b]) = w$ to belong to $\mathbb{P}(C)$, respectively, to $\mathbb{P}(C^{\perp})$, then

$$c_i = \sum_{w=d}^{d+i} \pi^{(w)}_{\mathbb{P}(C)} \binom{d+i}{w} (q-1)^{w-1} \text{ for } \forall 0 \leq i \leq g-1;$$

$$c_i = q^{i-g+1} \left[\sum_{w=d^{\perp}}^{n-d-i} \pi^{(w)}_{\mathbb{P}(C^{\perp})} \binom{n-d-i}{w} (q-1)^{w-1} \right], \forall g \leq i \leq g+g^{\perp}-2.$$

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$$c_i = \sum_{[a] \in \mathbb{P}(C)} \overline{\pi}_{[a]}^{(d+i)} \quad \text{for } \forall 0 \leq i \leq g-1;$$

$$c_i = q^{i-g+1} \left(\sum_{[b] \in \mathbb{P}(C^{\perp})} \overline{\pi}_{[b]}^{(n-d-i)} \right) \quad \text{for } \forall g \leq i \leq g+g^{\perp}-2.$$

Thank you for your attention!