Threshold Decoding for Disjunctive Group Testing

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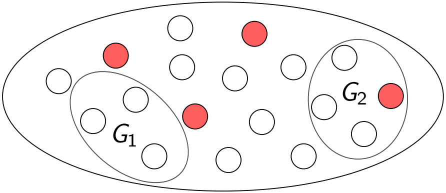
Group Testing Problem

Given a pool of t elements. $\mathcal{S}_{un} \subset [t]$ – a set of defective elements.

Group test: $G \subset [t]$. The test result: $y = \begin{cases} 1, & |G \cap \mathcal{S}_{un}| > 0, \\ 0, & |G \cap \mathcal{S}_{un}| = 0. \end{cases}$



$y_1 = 0$
$y_2 = 1$

Conventional Problem

Assumption: the number of defects $\leqslant s$.
Problem: to find all defective elements.

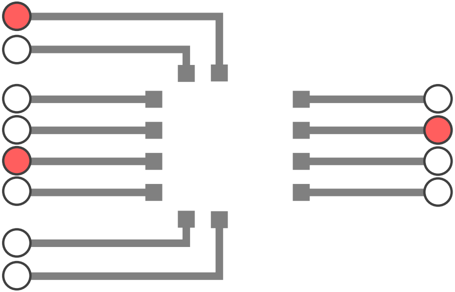
Threshold Problem

Problem: to decide whether $|\mathcal{S}_{un}| \leqslant s$ or $|\mathcal{S}_{un}| > s$.

Motivation of Threshold Problem

[1976, Malyutov]

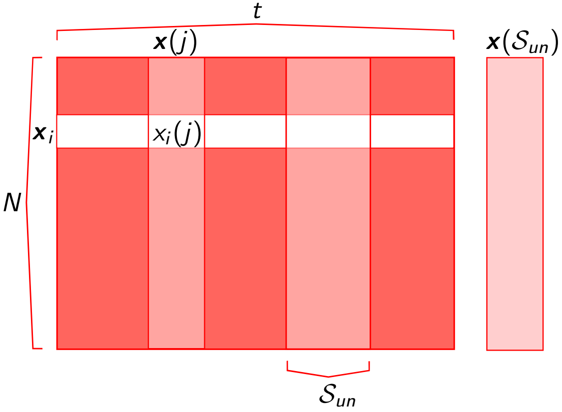
Given a complex electronic circuit of size t.
The circuit is not working properly if $> s$ elements are defective.
Such a circuit should be changed by a new similar circuit.



Testing time should be small.
We consider only nonadaptive algorithms.

Binary Code

N tests $\longleftrightarrow$ binary $(N \times t)$-matrix X (code X with codewords $\boldsymbol{x}(1), \ldots, \boldsymbol{x}(t)$)



$$x_i(j) = 1$$
$$\Longleftrightarrow$$
j-th element is in i-th test

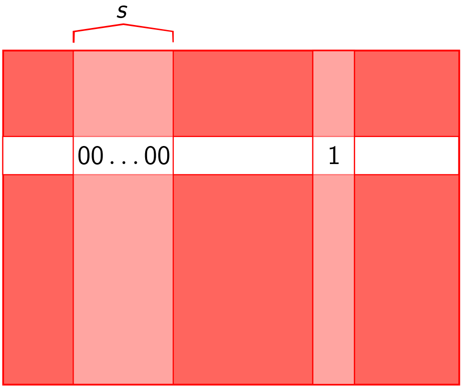
Result column:

$$\boldsymbol{x}(\mathcal{S}_{un}) = \bigvee_{j \in \mathcal{S}_{un}} \boldsymbol{x}(j)$$

Conventional Decoding Algorithm

We say that a binary column $\mathbf{u}$ covers a binary column $\mathbf{v}$ if $\mathbf{u}_i \geqslant \mathbf{v}_i, \forall\, i \in [N]$.

Definition. A code X is called a disjunctive s-code if the disjunctive sum of any s codewords of X does not cover any other codeword.



To find all $\leqslant s$ defects using disjunctive s-code, we find all codewords, which are covered by the result column.

Necessity and Sufficiency

Conventional Problem

Assumption: $|\mathcal{S}_{un}| \leqslant s$.
Problem: to find $\mathcal{S}_{un}$.

Threshold Problem

Problem: to decide whether
$|\mathcal{S}_{un}| \leqslant s$ or $|\mathcal{S}_{un}| > s$.

Proposition 1. A code X is disjunctive s-code $\Longrightarrow$ X solves conventional problem.

Proposition 2. A code X solves conventional problem $\Longrightarrow$ X is disjunctive $(s-1)$-code.

Proposition 3. A code X is disjunctive s-code $\Longrightarrow$ X solves threshold problem.

Proposition 4. A code X solves threshold problem $\Longrightarrow$ X is disjunctive s-code.

Hypothesis Testing

The null hypothesis
$H_0 \; : \; |\mathcal{S}_{un}| \leqslant s$

The alternative
$H_1 \; : \; |\mathcal{S}_{un}| \geqslant s+1$

The probability distribution of the random collection $\mathcal{S}_{un}$, is identified by vector $\boldsymbol{p} \triangleq (p_0, p_1, \ldots, p_t)$, $p_k \geqslant 0$, $\sum_{k=0}^{t} p_k = 1$, as follows:

$$\Pr\{\mathcal{S}_{un} = \mathcal{S}\} \triangleq \frac{p_{|\mathcal{S}|}}{\binom{t}{|S|}} \quad \text{for any subset} \quad \mathcal{S} \subset [t].$$

Hypothesis Testing

The null hypothesis
$H_0 \ : \ |\mathcal{S}_{un}| \leqslant s$

The alternative
$H_1 \ : \ |\mathcal{S}_{un}| \geqslant s+1$

Suppose we have some group testing procedure (code X) and some decision rule (DR).
Introduce maximal error probability

$$\varepsilon_s(\mathrm{DR}, \boldsymbol{p}, X) \triangleq \max \left\{ \Pr\{\text{accept } H_1 \big| H_0\}, \Pr\{\text{accept } H_0 \big| H_1\} \right\}.$$

Hypothesis Testing

The null hypothesis
$$H_0 \; : \; |\mathcal{S}_{un}| \leqslant s$$

The alternative
$$H_1 \; : \; |\mathcal{S}_{un}| \geqslant s+1$$

$$\varepsilon_s(\mathrm{DR}, \boldsymbol{p}, X) \triangleq \max \left\{ \Pr\{\text{accept } H_1 \big| H_0\} , \Pr\{\text{accept } H_0 \big| H_1\} \right\}.$$

Introduce universal error probability for the DR

$$\varepsilon_s^N(\mathrm{DR}, R) \triangleq \max_{\boldsymbol{p}} \left\{ \min_X \varepsilon_s(\mathrm{DR}, \boldsymbol{p}, X) \right\},$$

where the minimum is taken over all codes X of length N and size $t = \lfloor 2^{RN} \rfloor$.

Hypothesis Testing

The null hypothesis
$H_0 \; : \; |\mathcal{S}_{un}| \leqslant s$

The alternative
$H_1 \; : \; |\mathcal{S}_{un}| \geqslant s+1$

$$\varepsilon_s(\mathrm{DR}, \boldsymbol{p}, X) \triangleq \max \left\{ \Pr\{\text{accept } H_1 \big| H_0\}, \, \Pr\{\text{accept } H_0 \big| H_1\} \right\}.$$

$$\varepsilon_s^N(\mathrm{DR}, R) \triangleq \max_{\boldsymbol{p}} \left\{ \min_{X} \varepsilon_s(\mathrm{DR}, \boldsymbol{p}, X) \right\},$$

The error exponent

$$E_s(\mathrm{DR}, R) \triangleq \overline{\lim_{N\to\infty}} \frac{-\log_2 \varepsilon_s^N(\mathrm{DR}, R)}{N},$$

identifies the asymptotic behavior of α-level of significance for the decision rule, i.e.,

$$\alpha \triangleq 2^{-N\,[E_s(\mathrm{DR},R)+o(1)]}, \quad \text{if} \quad E_s(\mathrm{DR}, R) > 0, \quad N \to \infty.$$

Disjunctive Decision Rule (Previous Results)

The disjunctive decision rule (DDR) is based on the conventional decoding algorithm:

$$\begin{cases} \text{accept } \{H_0 : |\mathcal{S}_{un}| \leqslant s\} & \text{if } \boldsymbol{x}(\mathcal{S}_{un}) \text{ covers} \leqslant s \text{ codewords of } X, \\ \text{accept } \{H_1 : |\mathcal{S}_{un}| \geqslant s+1\} & \text{if } \boldsymbol{x}(\mathcal{S}_{un}) \text{ covers} \geqslant s+1 \text{ codewords of } X. \end{cases}$$

The maximum in universal error probability for the DDR is attained at the distribution vector

$$\boldsymbol{p} = (0, 0, \ldots, 0, \underset{s\uparrow}{1}, 0, \ldots, 0)$$

[2015, D'yachkov et al]

Theorem 1. If $R \geqslant \frac{1}{s}$, then $E_s(\mathrm{DDR}, R) = 0$.

Theorem 2. If $R < \frac{\ln 2}{s}$, then $E_s(\mathrm{DDR}, R) > 0$.

Threshold Decision Rule (New Results)

For a fixed parameter τ, $0 < \tau < 1$, introduce the τ-threshold decision rule:

$$\begin{cases} \text{accept } \{H_0 \,:\, |\mathcal{S}_{un}| \leqslant s\} & \text{if } |\boldsymbol{x}(\mathcal{S}_{un})| \leqslant \lfloor \tau N \rfloor, \\ \text{accept } \{H_1 \,:\, |\mathcal{S}_{un}| \geqslant s+1\} & \text{if } |\boldsymbol{x}(\mathcal{S}_{un})| \geqslant \lfloor \tau N \rfloor + 1. \end{cases}$$

The maximum in universal error probability for the DDR is attained at the distribution vector

$$\boldsymbol{p} = (0, 0, \ldots, 0, \tfrac{1}{2}, \tfrac{1}{2}, 0, \ldots, 0)$$

$$s \Big\uparrow$$

Theorem. The error exponent for τ-threshold decision rule satisfies the inequality:

$$E_s(\tau, R) \geqslant \underline{E}_s(\tau).$$

The optimal value of error exponent is not less than

$$\underline{E}_{\text{Thr}}(s) \triangleq \max_{0<\tau<1} \underline{E}_s(\tau, R) \geqslant \frac{\log_2 e}{4s^2}(1 + o(1)), \quad s \to \infty.$$

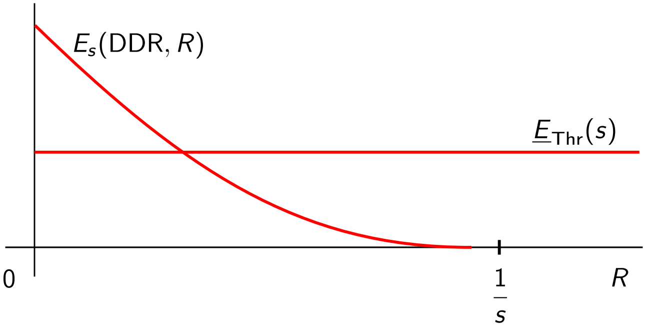
Comparison of Decision Rules (Error Exponent)

Disjunctive decision rule (DDR)

$$E_s(\mathrm{DDR}, R) = 0 \quad \text{if} \quad R \geqslant \frac{1}{s}.$$

Threshold decision rule (TDR)

$$E_s(\mathrm{TDR}, R) \geqslant \underline{E}_{\mathrm{Thr}}(s) \geqslant \frac{\log_2 e}{4s^2}.$$



Comparison of Decision Rules (Simulation)

The probability distribution vector $\boldsymbol{p}$ is such that $|\mathcal{S}_{un}|$ has binomial distribution with the mean $s + \frac{1}{2}$.

Code X of length N and size t is generated randomly from the ensemble of constant weight codes, i.e. every codeword of X has weight w.

$$s = 2, \quad t = 15$$

	Threshold decision rule			Disjunctive decision rule	
N	$\varepsilon_s(\mathrm{TDR}, \boldsymbol{p}, X)$	w	$\lfloor \tau N \rfloor$	$\varepsilon_s(\mathrm{DDR}, \boldsymbol{p}, X)$	w
5	**0.1366**	2	3	0.4780	2
8	**0.0824**	3	5	0.3610	2
10	**0.0744**	1	2	0.2390	3
12	**0.0440**	1	2	0.1220	3
14	**0.0349**	2	4	0.0537	3
15	0.0258	2	4	**0.0195**	3

Comparison of Decision Rules (Simulation)

The probability distribution vector $\boldsymbol{p}$ is such that $|\mathcal{S}_{un}|$ has binomial distribution with the mean $s + \frac{1}{2}$.

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$$s = 2, \quad t = 20$$

	Threshold decision rule			Disjunctive decision rule	
N	$\varepsilon_s(\text{TDR}, \boldsymbol{p}, X)$	w	$\lfloor \tau N \rfloor$	$\varepsilon_s(\text{DDR}, \boldsymbol{p}, X)$	w
5	**0.1398**	2	3	0.5356	2
8	**0.0897**	3	5	0.4169	2
10	**0.0897**	3	5	0.3008	3
12	**0.0580**	4	7	0.1979	3
14	**0.0429**	2	4	0.1214	4
15	**0.0324**	2	4	0.0792	4
17	**0.0246**	2	4	0.0449	4
18	**0.0139**	3	6	0.0290	5
20	0.0148	3	6	**0.0026**	5

Thank you for your attention!