Construction of some triple blocking sets in PG$(2, q)$

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1. Introduction

$\mathbb{F}_q$: the field of q elements

$\Sigma = \mathrm{PG}(2, q)$: projective plane over $\mathbb{F}_q$

- Σ forms a design $S(2, q+1, q^2+q+1)$.
- Σ is defined from $\mathbb{F}_q^3 \setminus \{(0,0,0)\}$ as

$$P(a,b,c) = Q(x,y,z) \text{ in } \Sigma$$

$$\Leftrightarrow (a,b,c) = \lambda(x,y,z) \text{ for some } \lambda \in \mathbb{F}_q \setminus \{0\}$$

for $P(a,b,c), Q(x,y,z) \in \mathbb{F}_q^3 \setminus \{(0,0,0)\}$.

A projectivity is a bijection $\tau: \Sigma \longrightarrow \Sigma$ induced by a 3×3 non-singular matrix over $\mathbb{F}_q$.

Two n-sets K_1 and K_2 in Σ are called projectively equivalent if there exists a projectivity τ such that $\tau(K_1) = K_2$.

For a given set K in Σ,

$\mathrm{Aut}(K) = \{\tau \in \mathrm{PGL}(3,q) \mid \tau(K) = K\}$ is the automorphism group of K.

A b-set B in PG$(2, q)$ is a (b, m)-blocking set if $m = \min\{|l \cap B| \ : \ l$ is a line$\}$.

$m = 1$: blocking set

$m = 2$: double blocking set

$m = 3$: triple blocking set

A line is a (trivial) $(q + 1, 1)$-blocking set in PG$(2, q)$.

A b-set B in PG$(2,q)$ is a (b,m)-blocking set if $m = \min\{|l \cap B| \ : \ l$ is a line$\}$.

$m = 1$: blocking set

$m = 2$: double blocking set

$m = 3$: triple blocking set

A $(b,1)$-blocking set containing no line is called non-trivial in PG$(2,q)$.

For a (b, m)-blocking set B in $\Sigma = \mathrm{PG}(2, q)$,

i-line: a line l with $i = |l \cap B|$

$b_i := |\{\text{line } l \mid |l \cap B| = i\}|$

The list of b_i's is the spectrum of B.

A set K of n points in Σ is an (n, r)-arc if $r = \max\{|l \cap K| \mid l\text{: a line}\}$.

An $(n, 2)$-arc is simply called an n-arc.

A set $\mathcal{L}$ of s lines in PG$(2, q)$ is an s-arc of lines if no 3 lines of $\mathcal{L}$ are concurrent.

For an (n, r)-arc K, the set $B = \Sigma \setminus K$ is a $(q^2+q+1-n, q+1-r)$-blocking set and vice versa.

Let m be an integer with $1 \leq m \leq q - 1$.

Problem 1.

(1) Find $b(m, q)$, the minimum value of b for which there exists a (b, m)-blocking set in $\mathrm{PG}(2, q)$.

(2) Classify (b, m)-blocking sets in $\mathrm{PG}(2, q)$ for $b = b(m, q)$ up to proj. equivalence.

B: a $(b, 1)$-blocking set in PG$(2, q)$

$b(m, q) = q + 1$, such B is a line.

Problem 2.

(1) Find $b(q)$, the minimum value of b for which there exists a non-trivial $(b, 1)$-blocking set in PG$(2, q)$.

(2) Classify non-trivial $(b, 1)$-blocking sets in PG$(2, q)$ for $b = b(q)$ up to projective equivalence.

Problem 2.

(1) Find $b(q)$, the minimum value of b for which there exists a non-trivial $(b, 1)$-blocking set in PG$(2, q)$.

Known results.

(1) $b(p) = 3(p + 1)/2$ for odd prime p.

(2) $b(q) = q + \sqrt{q} + 1$ if q is square.

(3) $b(q) = q + p^2 + 1$ if $q = p^3$ with p prime.

(4) A non-trivial $(b(q), 1)$-blocking set for square q is a Baer subplane PG$(2, \sqrt{q})$.

Problem 1 ($m = 2$).

(1) Find $b(2,q)$, the minimum value of b for which $^{\exists}(b,2)$-blocking set in PG$(2,q)$.

Known results.

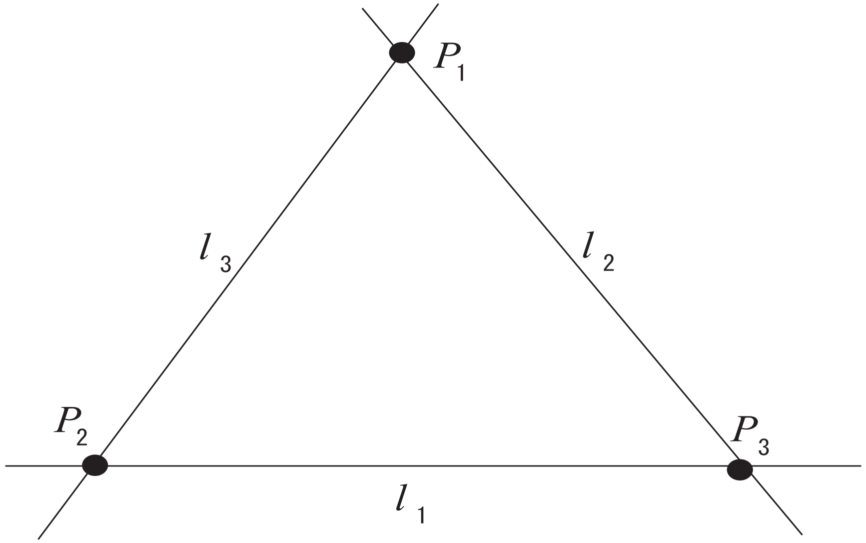
(1) $b(2,q) = 3q$ for $q \leq 8$.

(2) $b(2,q) \geq (5q+7)/2$ for $q = 11, 13, 17, 19$.

(3) $b(2,q) = 2(q + \sqrt{q} + 1)$ for square $q \geq 9$.

(4) A $(b,2)$-blocking set B containing a line satisfies $|B| \geq 3q$.

For a 3-arc of lines $\{l_1, l_2, l_3\}$,
$B = l_1 \cup l_2 \cup l_3$ forms a $(3q, 2)$-blocking set.



- $|\ell \cap B| = 2$ iff ℓ contains one of P_1, P_2, P_3.
- Spec: $(b_2, b_3, b_{q+1}) = (3(q-1), (q-1)^2, 3)$

Known results for $m = 3$.

A $(b, 3)$-blocking set B in PG$(2, q)$ satisfies

(1) $b \geq 4q$ for odd q if B contains a line.

(2) $b \geq 4q - 1$ for even q if B contains a line.

(3) $b \geq 3(q + \sqrt{q} + 1)$ if B contains no line.

(4) $b(3, q) = 4q$ for $q = 5, 7, 9$.

(5) $b(3, 8) = 31$.

(6) $b(3, q) = 3(q + \sqrt{q} + 1)$ for odd square $q > 121$.

2. A conic in PG$(2, q)$

quadric: a curve in PG$(2, q)$ with homogeneous quadratic equation in 3 variables.

A non-singular quadric C is a conic.

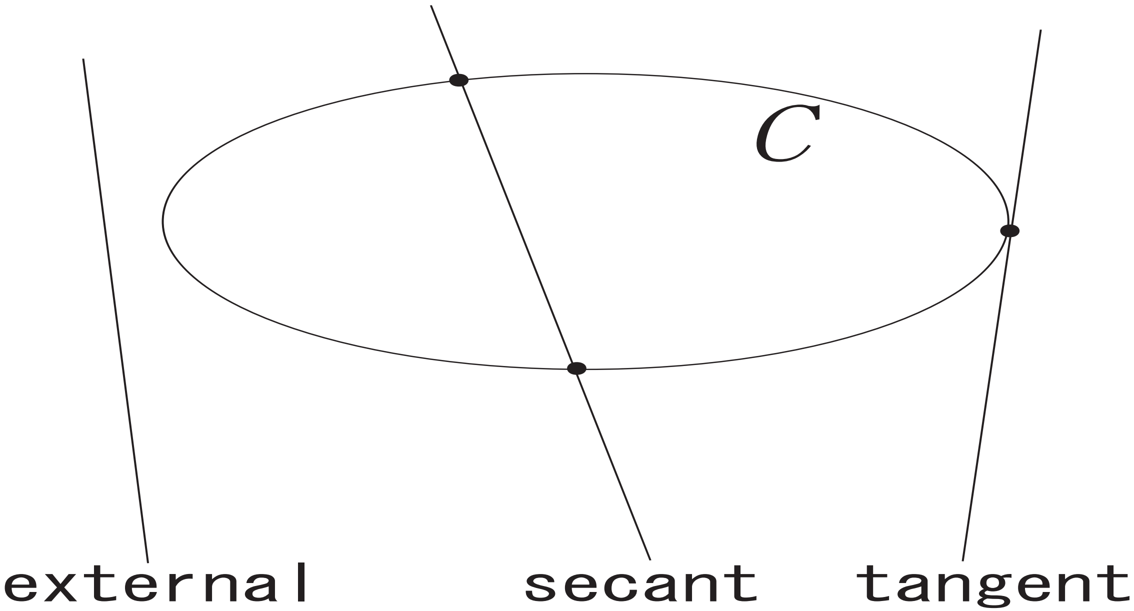
Normally, $C = \{(1, t, t^2) \ : \ t \in \mathbb{F}_q\} \cup \{(0, 0, 1)\}$.

Thm A. (B. Segre 1955)

Every $(q + 1)$-arc in PG$(2, q)$ with q odd is a conic.

Let C be a conic.

A line l is called external, tangent or secant to C if $|C \cap l| = 0$, 1 or 2, respectively.



Thm B.

When q is odd, the $q+1$ tangents of a conic C in PG$(2,q)$ form a $(q+1)$-arc of lines.

Thm C.

When q is even, the $q+1$ tangents of a conic C in PG$(2,q)$ are concurrent at the point N called the nucleus of C.

Hence, $C \cup \{N\}$ is a $(q+2)$-arc.

Known results for $m = 3$.

A $(b, 3)$-blocking set B in PG$(2, q)$ satisfies

(1) $b \geq 4q$ for odd q if B contains a line.

(2) $b \geq 4q - 1$ for even q if B contains a line.

(3) $b \geq 3(q + \sqrt{q} + 1)$ if B contains no line.

(4) $b(3, q) = 4q$ for $q = 5, 7, 9$.

(5) $b(3, 8) = 31$.

(6) $b(3, q) = 3(q + \sqrt{q} + 1)$ for odd square
$q > 121$.

3. **Construction of optimal triple blocking sets**

Our aim is to construct new optimal triple blocking sets containing a line in PG$(2, q)$.

Notation.

$[abc]$ (or $[a, b, c]$) denotes the line

$\{(x, y, z) \in \mathrm{PG}(2, q) \;:\; ax + by + cz = 0\}$.

For two points P and Q,

$\langle P, Q\rangle$ denotes the line through P and Q.

Thm 1 (Hill-Mason 1981).

For odd $q \geq 5$, let

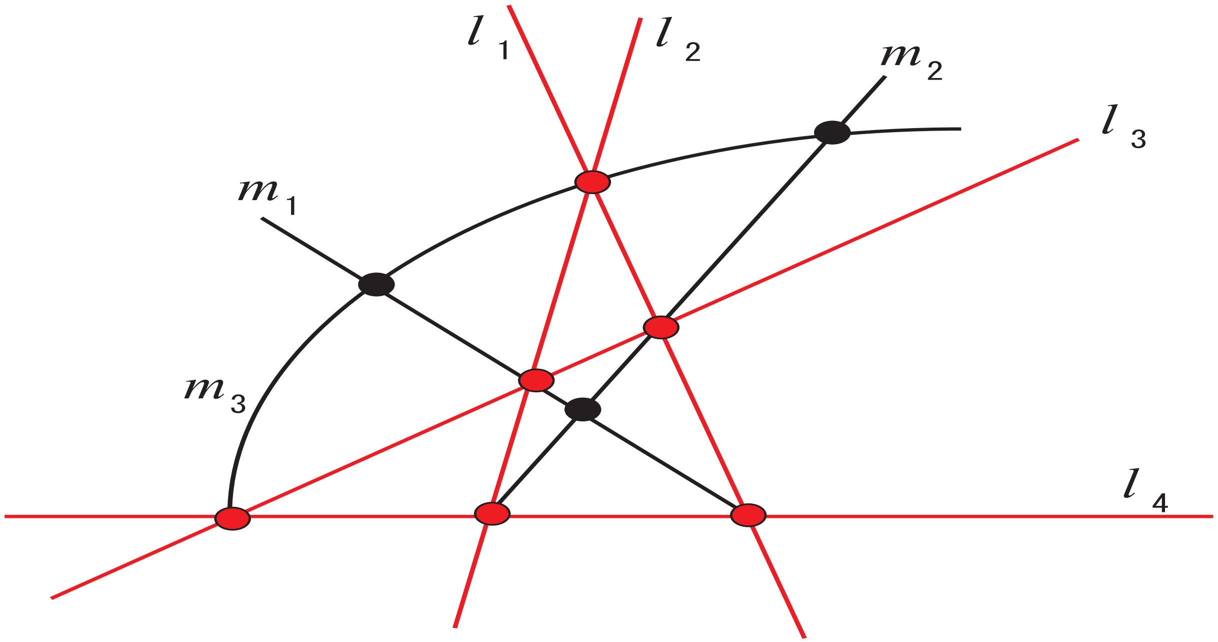
$$B = l_1 \cup l_2 \cup l_3 \cup l_4 \cup \{P, Q\},$$

consisting of the lines $l_1 = [100]$, $l_2 = [010]$, $l_3 = [001]$, $l_4 = [111]$ and the points $P = (-1, 1, 1)$, $Q = (1, -1, 1)$.

Then, B is a $(4q, 3)$-blocking set with spec.

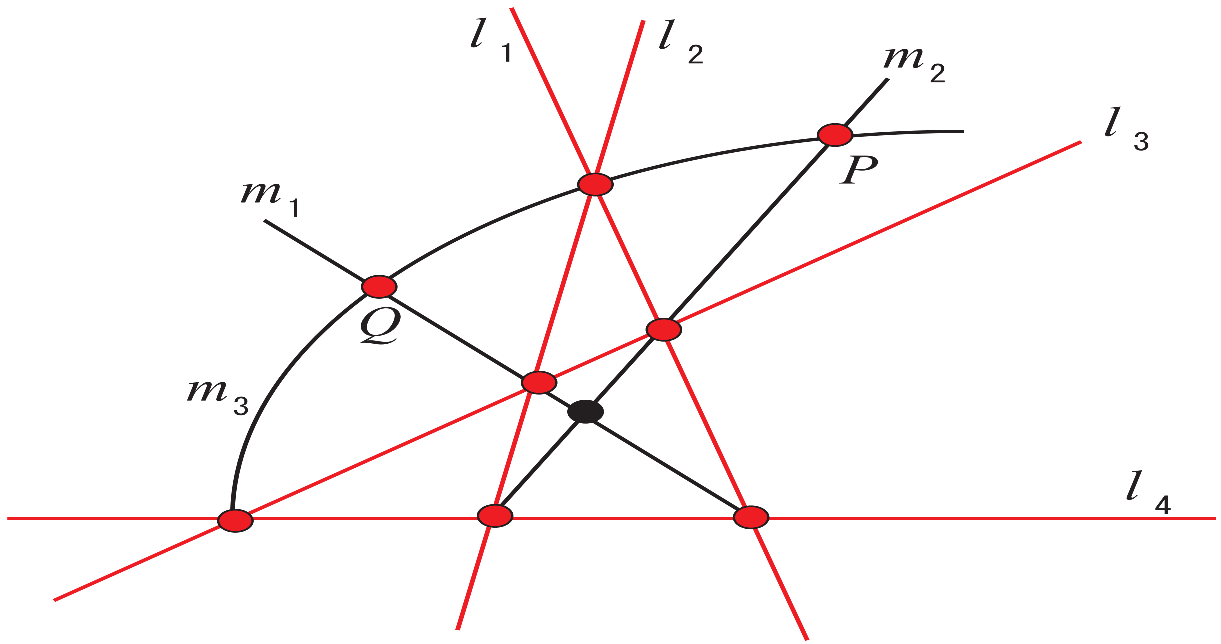
$$(b_3, b_4, b_5, b_{q+1})$$
$$= (6q - 14, q^2 - 7q + 17, 2q - 6, 4).$$

$$L = l_1 \cup l_2 \cup l_3 \cup l_4$$



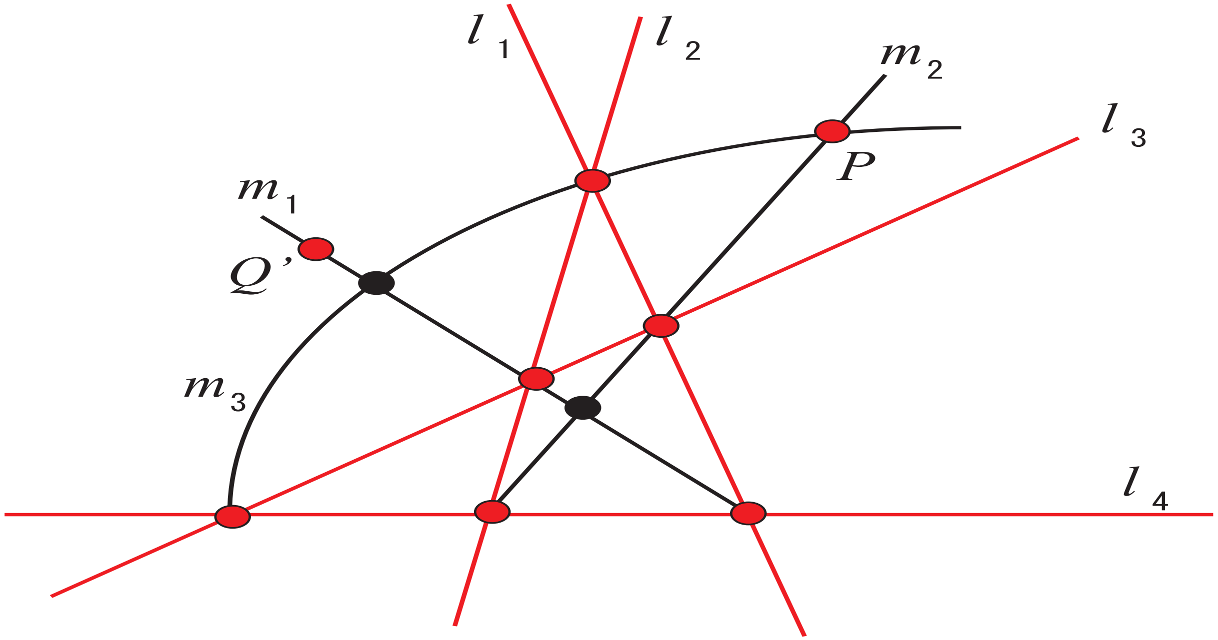
- m_1, m_2, m_3 are the 2-lines for L.

$B = l_1 \cup l_2 \cup l_3 \cup l_4 \cup \{P, Q\}$ (Hill-Mason)



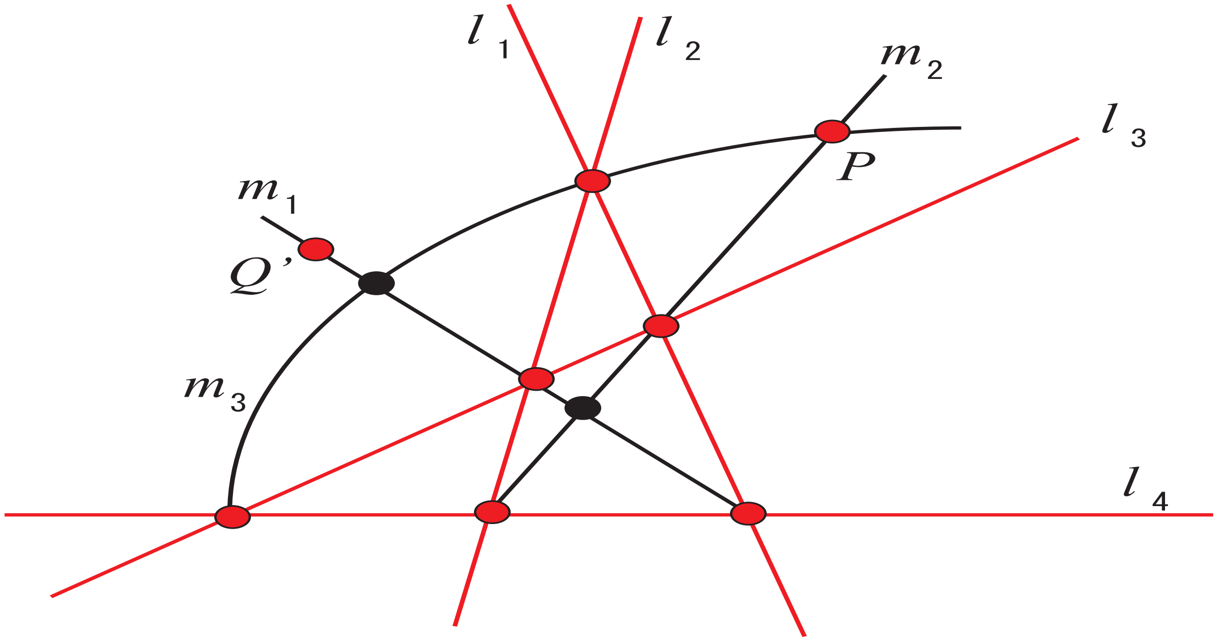
- B is a $(4q, 3)$-blocking set.

$$B' = l_1 \cup l_2 \cup l_3 \cup l_4 \cup \{P, Q'\}$$



Thm 2. B' is also a $(4q, 3)$-blocking set which is not projectively equivalent to B.

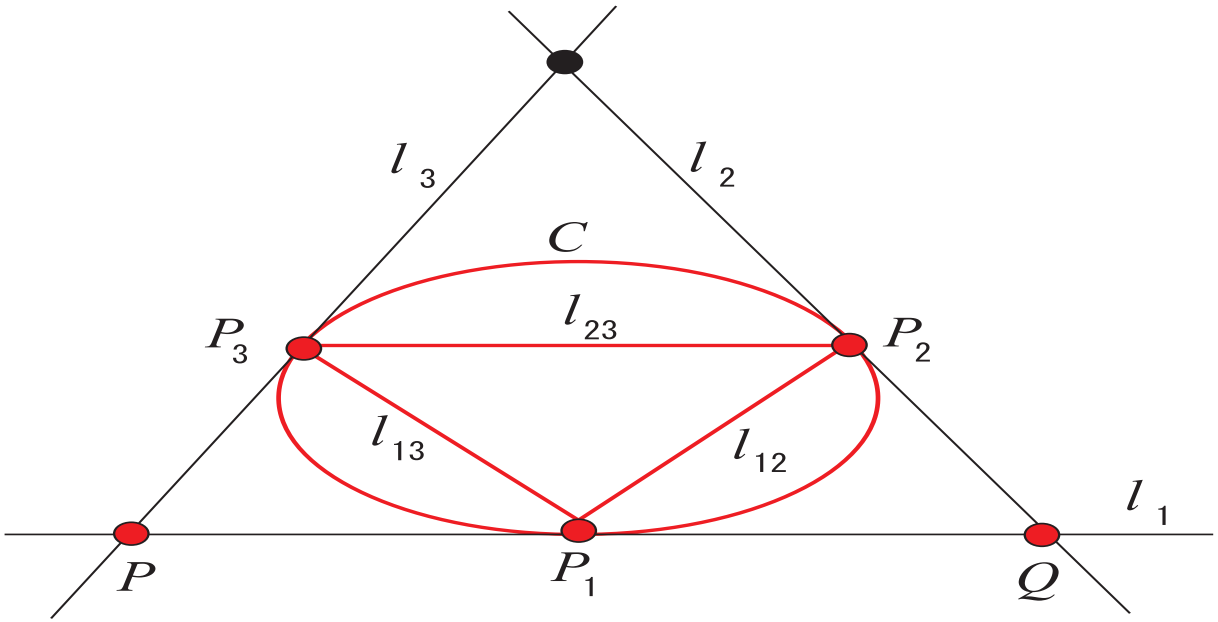
$$B' = l_1 \cup l_2 \cup l_3 \cup l_4 \cup \{P, Q'\}$$



- B' has spec. $(b_3, b_4, b_5, b_6, b_{q+1})$
  $= (6q - 15, q^2 - 7q + 20, 2q - 9, 1, 4)$.

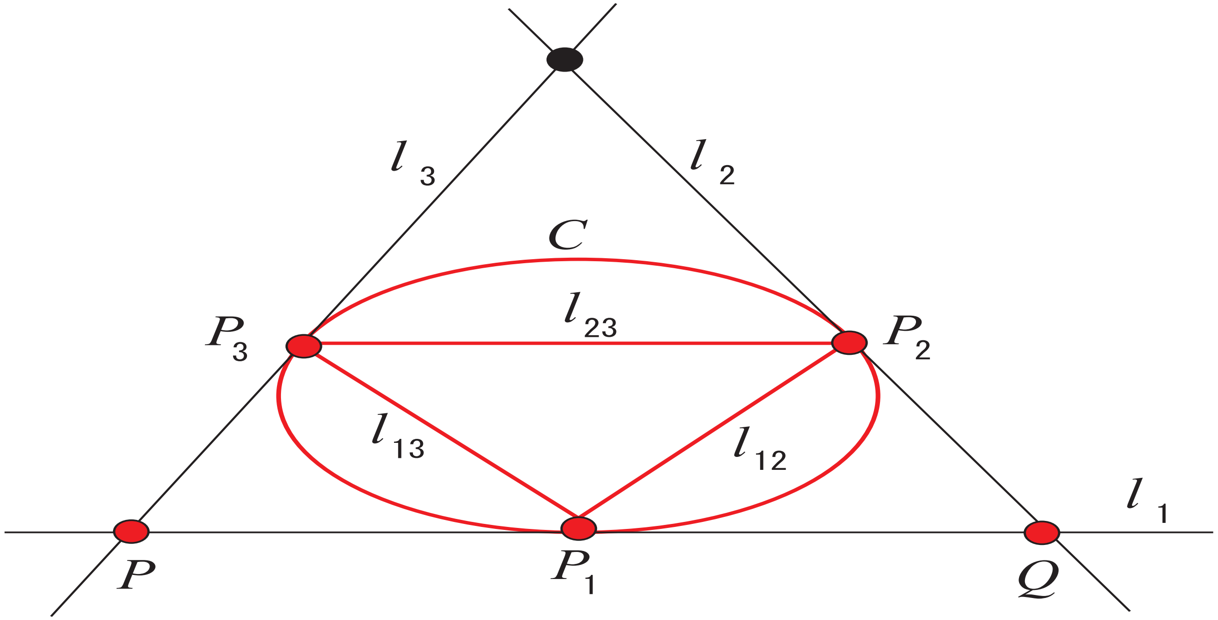
Thm 3. For odd $q \geq 5$, let C be a conic in PG$(2, q)$. For any three points P_1, P_2, P_3 in C, let l_i be the tangent of $\mathcal{C}$ through P_i and l_{ij} be the secant of C through P_i and P_j, and let $P_{ij} = l_i \cap l_j$ for $1 \leq i \leq j \leq 3$. Take any two points P and Q from the three points P_{12}, P_{23}, P_{13}, and let $B = C \cup l_{12} \cup l_{23} \cup l_{13} \cup \{P, Q\}$. Then, B is a $(4q, 3)$-blocking set.

$$B = C \cup l_{12} \cup l_{13} \cup l_{23} \cup \{P, Q\}$$



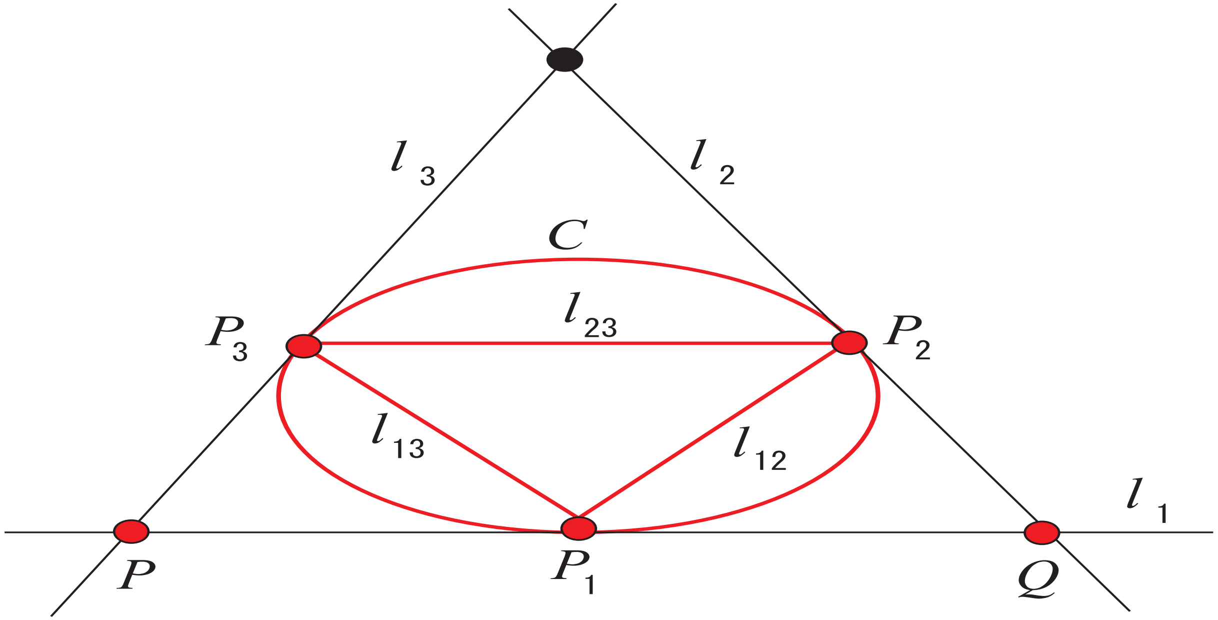
- $|B| = q + 1 + 3(q + 1) - 2 \cdot 3 + 2 = 4q$

$$B = C \cup l_{12} \cup l_{13} \cup l_{23} \cup \{P, Q\}$$



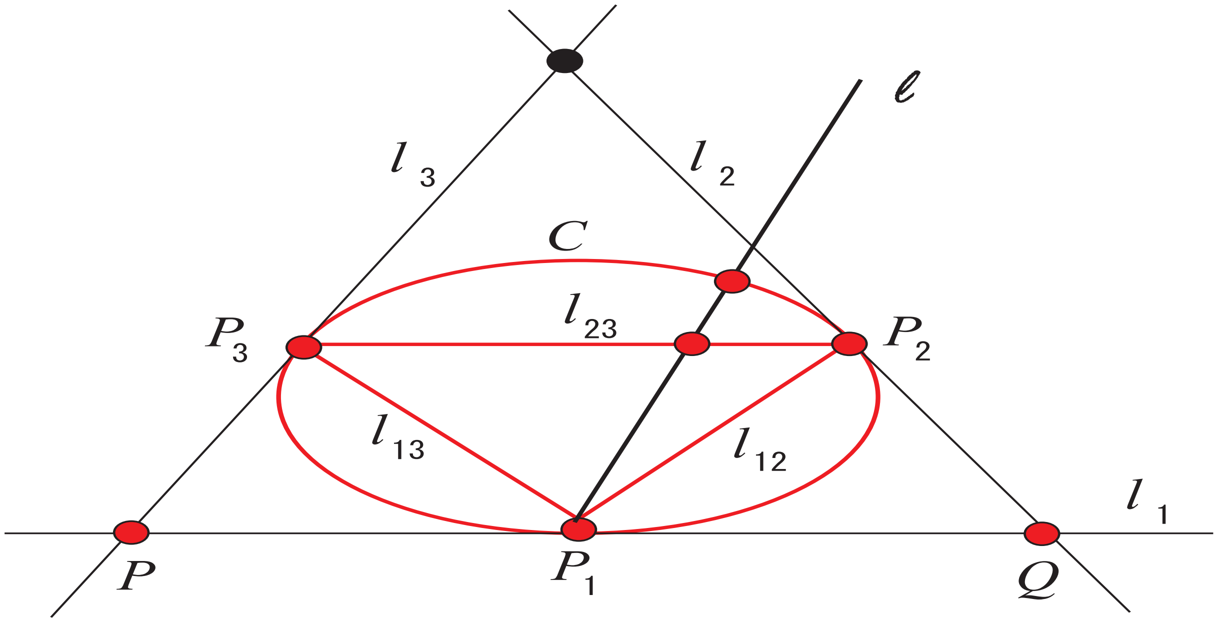
- $|\ell \cap (l_{12} \cup l_{13} \cup l_{23})| \geq 2$ for any line ℓ

$$B = C \cup l_{12} \cup l_{13} \cup l_{23} \cup \{P, Q\}$$



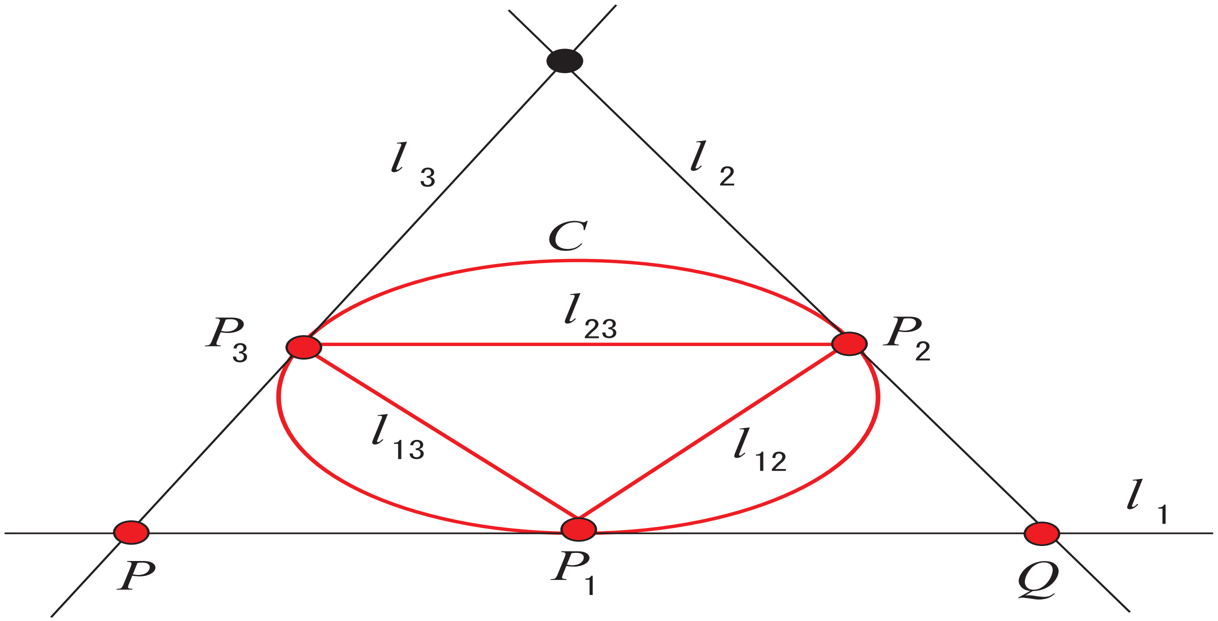
- If $|\ell \cap (l_{12} \cup l_{13} \cup l_{23})| = 2$ for some line ℓ, then ℓ contains one of P_1, P_2, P_3.

$$B = C \cup l_{12} \cup l_{13} \cup l_{23} \cup \{P, Q\}$$



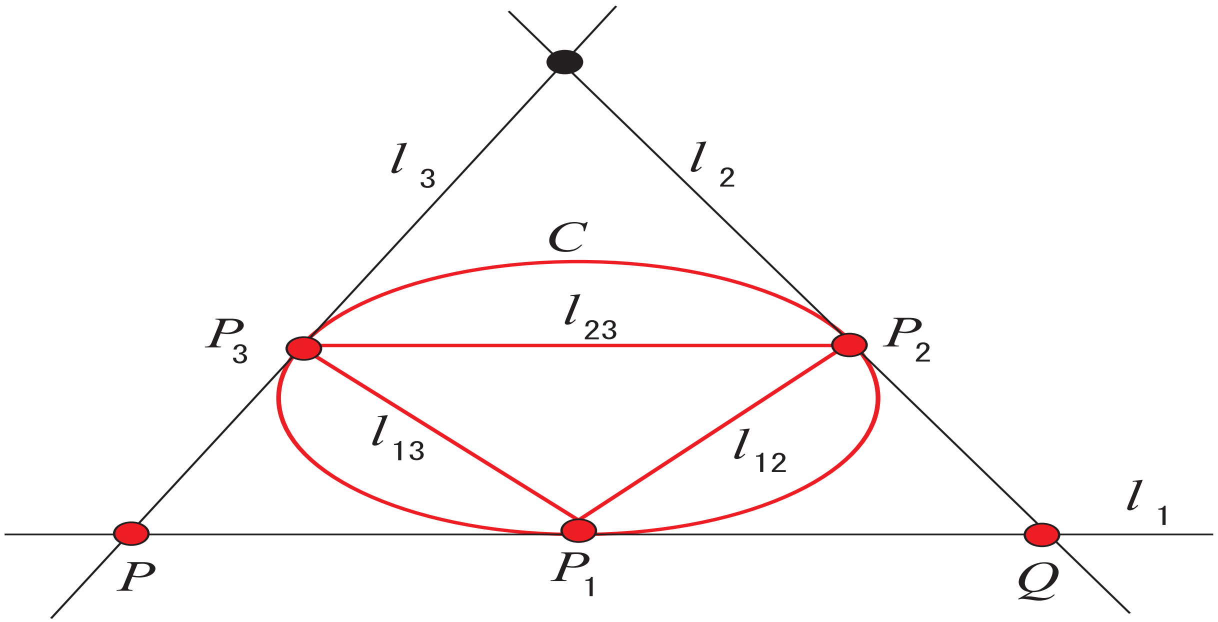
- If ℓ contains one of P_1, P_2, P_3 and if ℓ is a secant, then $|\ell \cap B| = 3$.

$$B = C \cup l_{12} \cup l_{13} \cup l_{23} \cup \{P, Q\}$$



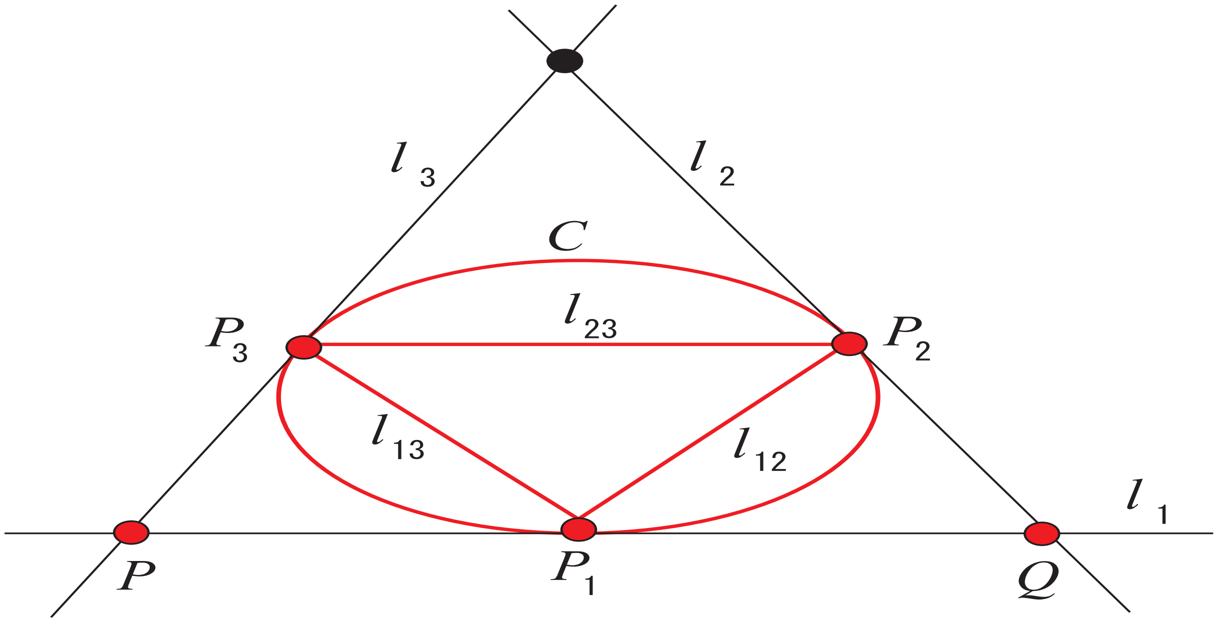
- If ℓ contains one of P_1, P_2, P_3 and if ℓ is a tangent, then $|\ell \cap B| = 3$ or 4.

$$B = C \cup l_{12} \cup l_{13} \cup l_{23} \cup \{P, Q\}$$



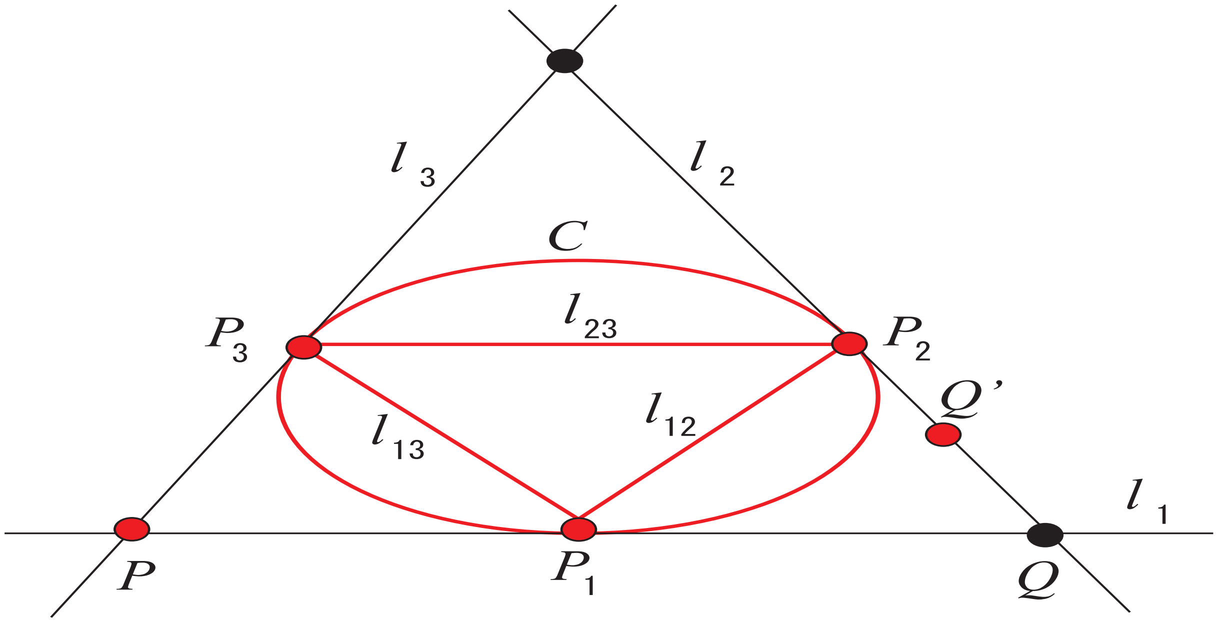
- B is a $(4q, 3)$-blocking set.

$$B = C \cup l_{12} \cup l_{13} \cup l_{23} \cup \{P, Q\}$$



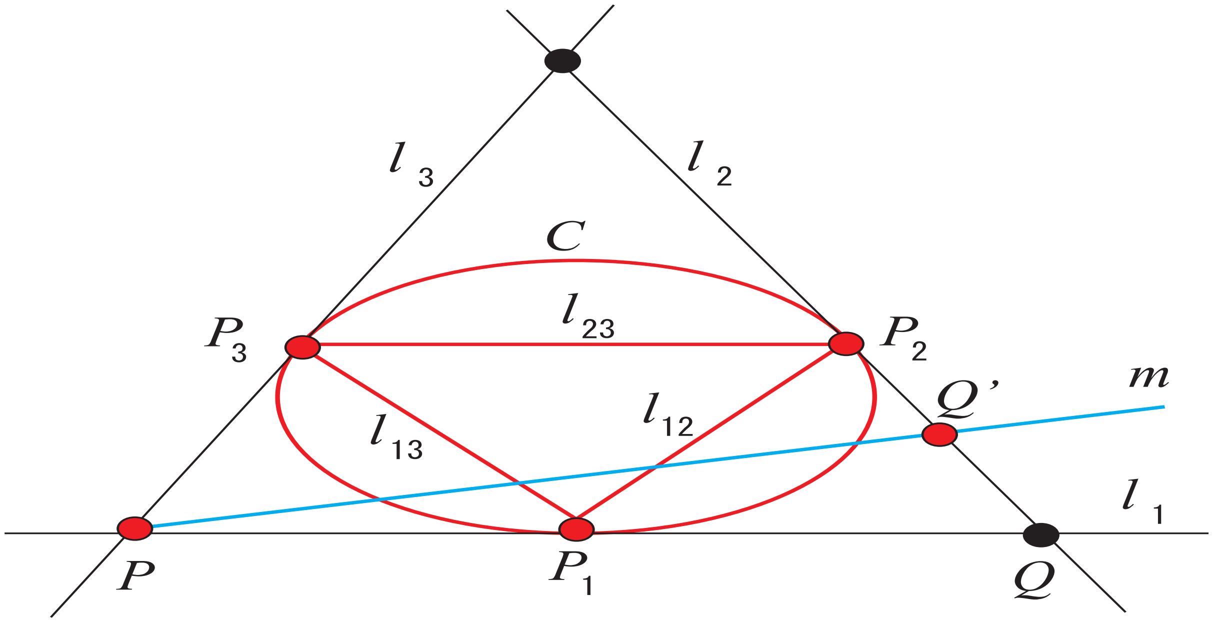
- B has spectrum $(b_3, b_4, b_5, b_6, b_{q+1})$
  $= (\frac{(q+5)(q-2)}{2}, 2q, \frac{(q-3)(q-4)}{2}, q-3, 3).$

$$B' = C \cup l_{12} \cup l_{13} \cup l_{23} \cup \{P, Q'\}$$



- B' is also a $(4q, 3)$-blocking set if $Q' \in l_2 \setminus \{P_2, Q, l_{13} \cap l_2\}$.

$$B' = C \cup l_{12} \cup l_{13} \cup l_{23} \cup \{P, Q'\}$$



- B' is also a $(4q, 3)$-blocking set if $Q' \in l_2 \setminus \{P_2, Q, l_{13} \cap l_2\}$. Let $m = \langle P, Q' \rangle$.

Thm 4. $B' = C \cup l_{12} \cup l_{13} \cup l_{23} \cup \{P, Q'\}$ has spectrum

(1) $(b_3, b_4, b_5, b_6, b_{q+1})$
$= (\frac{(q+5)(q-2)}{2}, 2q, \frac{(q-3)(q-4)}{2}, q-3, 3)$

(2) $(b_3, b_4, b_5, b_6, b_7, b_{q+1})$
$= (\frac{(q+5)(q-2)}{2}, 2q-1, \frac{q^2-7q+18}{2}, q-6, 1, 3)$

(3) $(b_3, b_4, b_5, b_6, b_{q+1})$
$= (\frac{q^2+3q-8}{2}, 2q-3, \frac{q^2-7q+18}{2}, q-4, 3)$

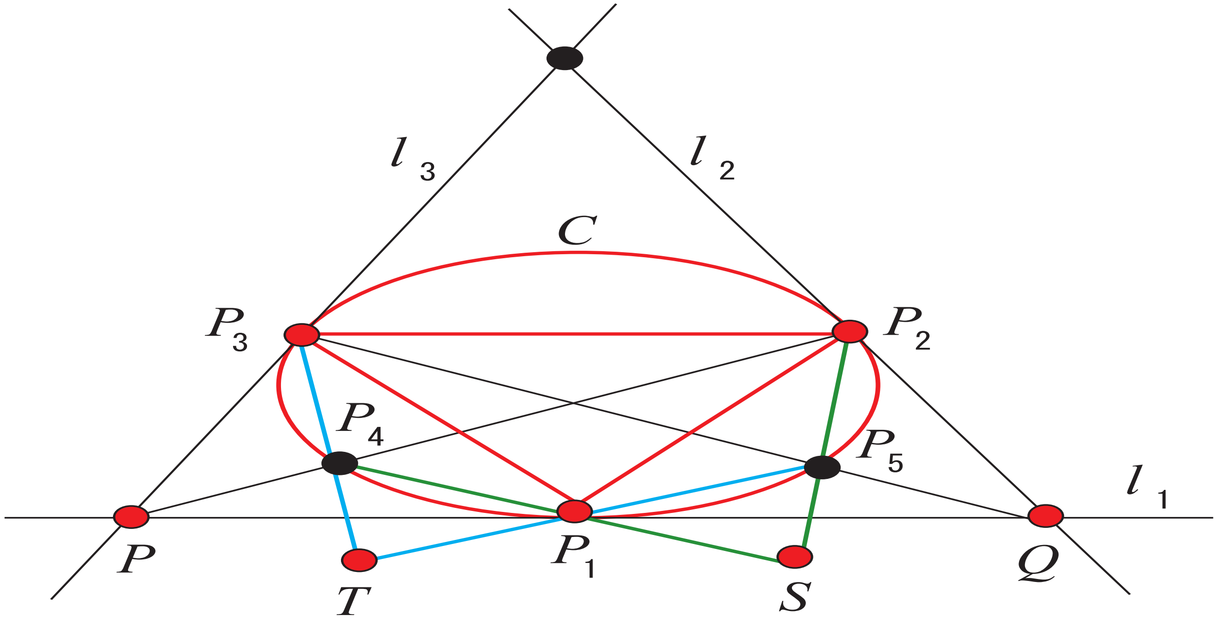
if m is a tangent, a secant or an external line, respectively.

Thm 5. Let $q = p^h \geq 7$ with odd prime $p \neq 3$. Under the conditions of Thm 3, let C be the conic $\{(1, a, a^2) \mid a \in \mathbb{F}_q\} \cup \{(0, 0, 1)\}$ and take $P_1 = (1, 1, 1)$, $P_2 = (0, 0, 1)$, $P_3 = (1, 0, 0)$, $P_4 = (1, 2^{-1}, 2^{-2})$, $P_5 = (1, 2, 2^2)$, $S = \langle P_1, P_4 \rangle \cap \langle P_2, P_5 \rangle$, $T = \langle P_1, P_5 \rangle \cap \langle P_3, P_4 \rangle$. Then, $B_1 = (B \backslash \{P_4, P_5\}) \cup \{S, T\}$ is a $(4q, 3)$-blocking set, which is not projectively equivalent to any blocking set in Thms 1-4.

Note. $P_4 = P_5$ iff $p = 3$.

$$B = C \cup l_{12} \cup l_{13} \cup l_{23} \cup \{P, Q\}$$

$$B_1 = (B \setminus \{P_4, P_5\}) \cup \{S, T\}$$



- B_1 is also a $(4q, 3)$-blocking set.

From the above thms, we get the following.

Corollary 1. There exist at least six projectively inequivalent $(4q, 3)$-blocking sets containing a line in PG$(2, q)$ for $q = p^h \geq 7$ with odd prime $p \neq 3$.

Corollary 2. There exist at least six projectively inequivalent $(q^2 - 3q + 1, q - 2)$-arcs in PG$(2, q)$ for $q = p^h \geq 7$ with odd prime $p \neq 3$.

Thm 6 (Hill-Mason 1981).

For even $q \geq 4$, let

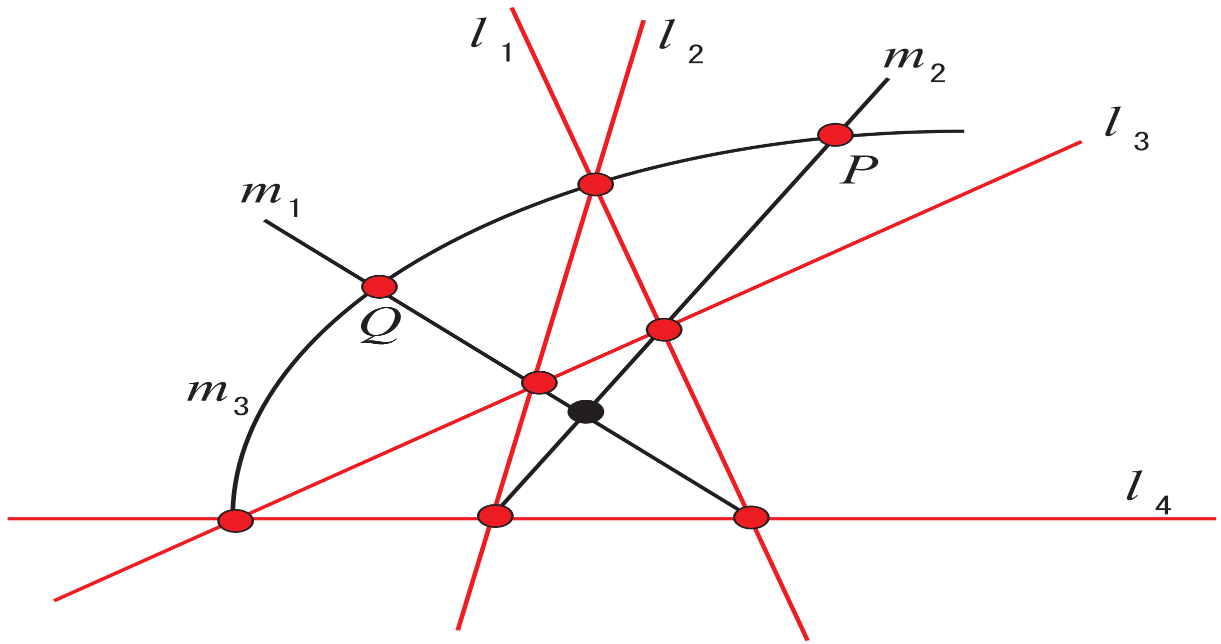
$$B = l_1 \cup l_2 \cup l_3 \cup l_4 \cup \{P\},$$

consisting of the lines $l_1 = [100]$, $l_2 = [010]$, $l_3 = [001]$, $l_4 = [111]$ and the point $P = (1, 1, 1)$.

Then, B is a $(4q - 1, 3)$-blocking set with spec. (b_3, b_4, b_5, b_{q+1})

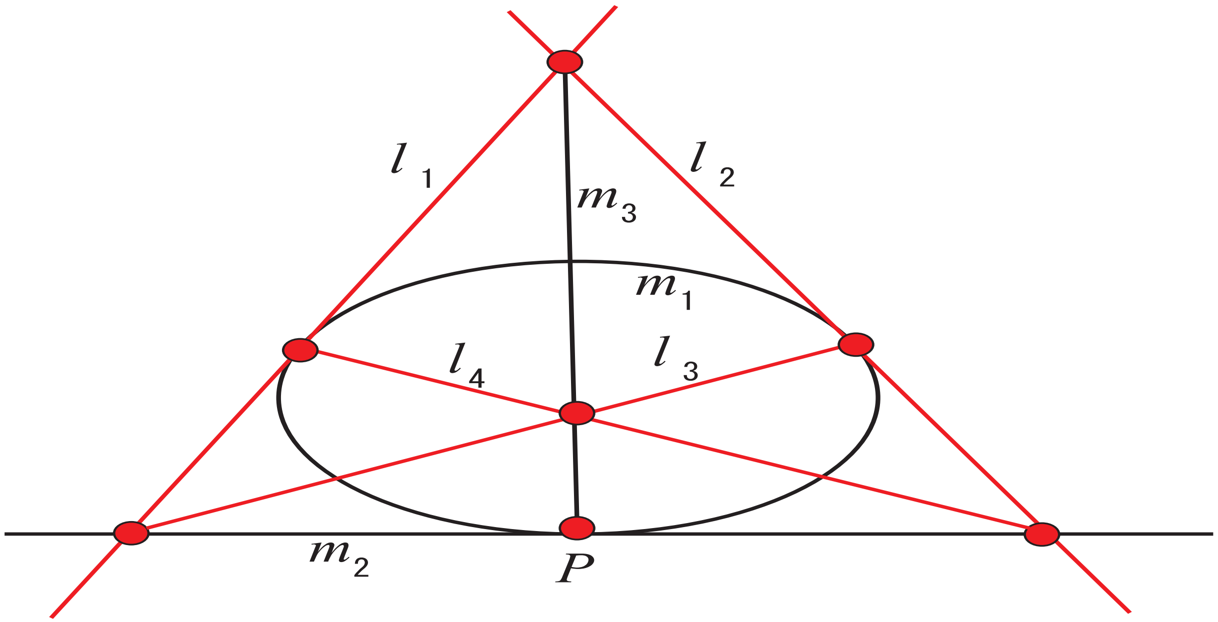
$$= (6q - 9, q^2 - 6q + 8, q - 2, 4).$$

For odd $q \geq 5$, $B = l_1 \cup l_2 \cup l_3 \cup l_4 \cup \{P, Q\}$



- B is a $(4q, 3)$-blocking set.

For even $q \geq 4$, $B = l_1 \cup l_2 \cup l_3 \cup l_4 \cup \{P\}$



- B is a $(4q-1, 3)$-blocking set.

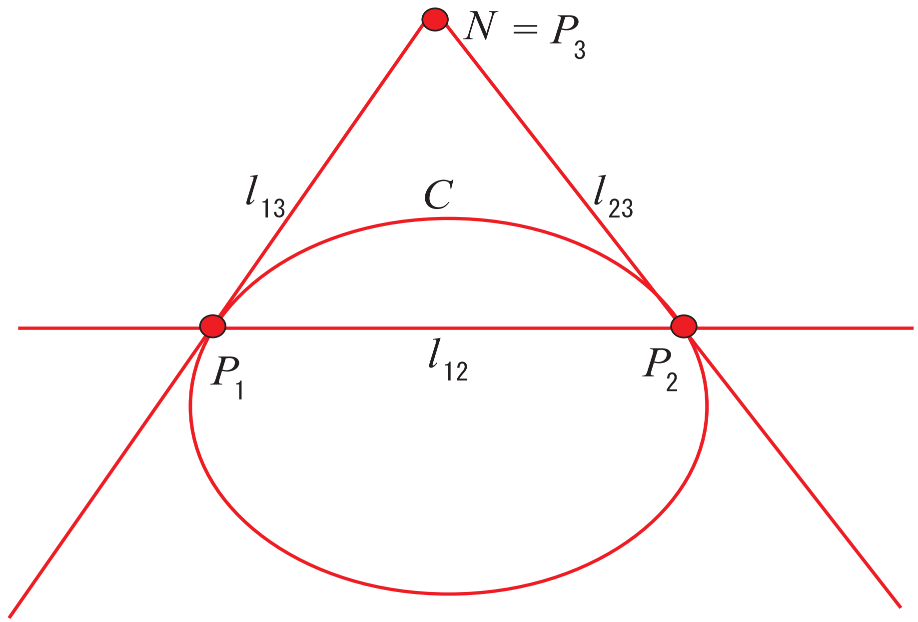
Thm 7. For even $q \geq 8$, let C be a conic in PG$(2,q)$ with nucleus N. For any points P_1, P_2, P_3 in $C \cup \{N\}$ with $P_1, P_2 \in C$, let $l_{ij} = \langle P_i, P_j \rangle$ for $1 \leq i < j \leq 3$. Then,

(1) $C \cup l_{12} \cup l_{23} \cup l_{13}$ is a $(4q-1,3)$-blocking set with $|\mathrm{Aut}(B)| = 2(q-1)$ if $P_3 = N$,

(2) $C \cup l_{12} \cup l_{23} \cup l_{13} \cup \{N\}$ is a $(4q-1,3)$-blocking set with $|\mathrm{Aut}(B)| = 6$ if $P_3 \neq N$.

$N = P_3, \; B = C \cup l_{12} \cup l_{13} \cup l_{23}$

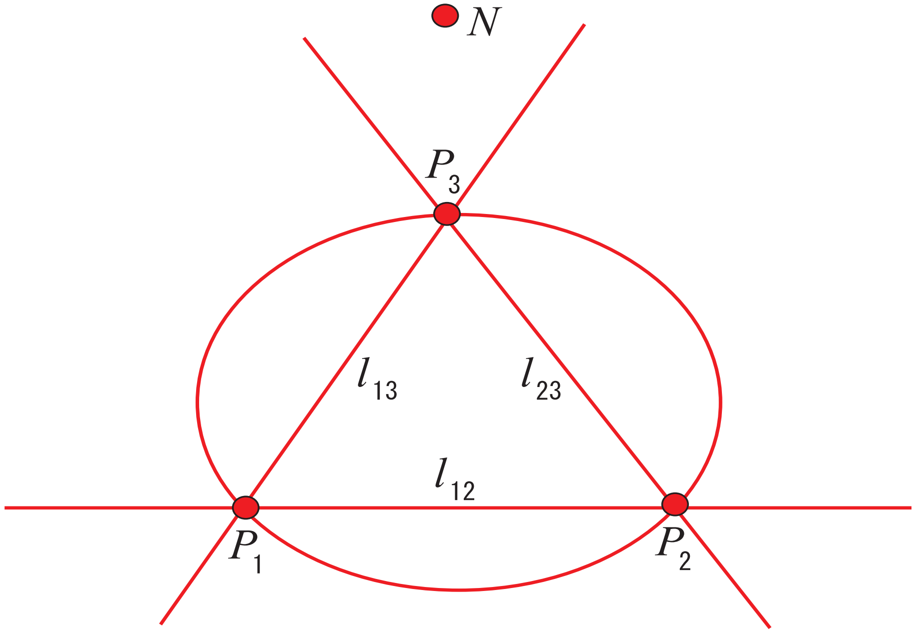
$|B| = q + 1 + 3(q + 1) - 2 - 2 - 1 = 4q - 1$



spec: $(b_3, b_5, b_{q+1}) = (\frac{(q+6)(q-1)}{2}, \frac{(q-1)(q-2)}{2}, 3)$

$$N = P_3,\ B = C \cup l_{12} \cup l_{13} \cup l_{23} \cup \{N\}$$

$$|B| = q + 1 + 3(q+1) - 2 - 2 - 2 + 1 = 4q - 1$$



spec: $(b_3, b_5, b_{q+1}) = (\frac{(q+6)(q-1)}{2}, \frac{(q-1)(q-2)}{2}, 3)$

From Thms 6 and 7, we get the following.

Corollary 3. There exist at least three projectively inequivalent $(4q-1,3)$-blocking sets containing a line in $\mathrm{PG}(2,q)$ for even $q \geq 8$.

Corollary 4. There exist at least six projectively inequivalent $(q^2-3q+2,q-2)$-arcs in $\mathrm{PG}(2,q)$ for even $q \geq 8$.

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Corollary 3. There exist at least three projectively inequivalent $(4q-1,3)$-blocking sets containing a line in $\mathrm{PG}(2,q)$ for even $q \geq 8$.

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Thank you for your attention!

References

[1] S. Ball, On the size of a triple blocking set in $PG(2,q)$, *European J. Combin.* **17** (1996) 427–435.

[2] A. Betten, E.J. Cheon, S.J. Kim, T. Maruta, The classification of $(42,6)_8$-arcs, *Adv. Math. Commun.* **5** (2011) 209–223.

[3] A. Blokhuis, On multiple nuclei and a conjecture of Lunelli and Sce, *Bull. Belg. Math. Soc.* **3** (1994) 349–353.

[4] A.A. Bruen, Polynomial multiplicities over finite fields and intersection sets, *J. Combin. Theory Ser. A* **60** (1992) 19–33.

[5] R. Hill, J.R.M. Mason, On (k,n)-arcs and the falsity of the Lunelli-Sce conjecture, in: London Math. Soc. Lecture Note Series 49, CUP, (1981), 153–168.

[6] J.W.P. Hirschfeld, Projective Geometries over Finite Fields 2nd ed., Clarendon Press, Oxford, 1998.