On maximal antipodal spherical codes with few distances

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Antipodal spherical codes

- $C \subset \mathbb{S}^{n-1}$, $|C| < \infty$, – spherical code
- If $C = -C$, then C is called antipodal
- **Problem**: Given dimension n, find the maximum possible cardinality of an antipodal code which under certain restrictions (for example, all inner products in $\{-1\} \cup [-s, s]$; or in $\{-1, \pm s\}$, etc.)
- Distance distribution with respect to $x \in C$ – the system $(A_t(x) : t \in [-1, 1), \exists y \in C, \langle x, y \rangle = t)$, where

$$A_t(x) = |\{y \in C : \langle x, y \rangle = t\}|.$$

Obvious properties: $A_{-1}(x) = 1$ for every $x \in C$, $A_t(x) = A_{-t}(x)$ for every $t \in (-1, 1)$ and every $x \in C$

LP bounds (1)

- For fixed dimension n, the Gegenbauer polynomials are defined by $P_0^{(n)} = 1$, $P_1^{(n)} = t$ and the three-term recurrence relation

$$(i+n-2)P_{i+1}^{(n)}(t) = (2i+n-2)tP_i^{(n)}(t) - iP_{i-1}^{(n)}(t)$$

  for $i \geq 1$

- If $f(t) \in \mathbb{R}[t]$ is a real polynomial of degree k, then $f(t)$ can be uniquely expanded in terms of the Gegenbauer polynomials as $f(t) = \sum_{i=0}^{k} f_i P_i^{(n)}(t)$

LP bounds (2)

- We use the identity

$$|C|f(1) + \sum_{x,y\in C, x\neq y} f(\langle x,y\rangle) = |C|^2 f_0 + \sum_{i=1}^{k} f_i M_i \qquad (1)$$

  as a source of estimations by polynomial techniques. Here $M_i := \frac{1}{r_i}\sum_{j=1}^{r_i}\left(\sum_{x\in C} Y_{ij}(x)\right)^2$ is the i-th moment of C, the functions $\{Y_{i,j},\ j=1,2,\ldots,r_i\}$, are the so-called spherical harmonics of degree i, and $r_i = \binom{n+i-3}{n-2}\frac{2i+n-2}{i}$

- C is antipodal iff $M_i = 0$ for every odd i. Further, a code C is a spherical τ-design if and only if its moments satisfy $M_i = 0$ for every positive integer $i \leq \tau$

Antipodal codes with inner products -1 and $\pm s$ (1)

- $C \subset \mathbb{S}^{n-1}$ – antipodal, $M = |C|$, C has inner products -1 and $\pm s$ (i.e C defines a system equiangular lines). Well known – if $M > 2n$ then $s = \frac{1}{2\ell+1}$, where ℓ is a positive integer. Denote by $M_{2\ell+1}(n)$ the maximum possible size of such C.
  LP bounds for equiangular lines were obtained by Barg and Yu (arxiv.org/abs/1311.3219). A. Barg, W.-H. Yu, New bounds on equiangular lines, in Discrete Geometry and Algebraic Combinatorics, A. Barg and O. Musin, eds., (Contemporary Mathematics, vol. 625), Amer. Math. Soc., Providence, RI, 2014, 111-121.

Theorem

(Barg, Yu) If $P_{2k}^{(n)}(\frac{1}{2\ell+1}) < 0$, *then* $M_{2\ell+1}(n) \leq 2 - \frac{2}{P_{2k}^{(n)}(\frac{1}{2\ell+1})}$.

Proof. Set $f(t) = P_{2k}^{(n)}(t)$ in (1). ☐

Antipodal codes with inner products -1 and $\pm s$ (2)

- For $k=1$ we have $P_2^{(n)}(t)=\frac{nt^2-1}{n-1}$ and therefore

$$M_{2\ell+1}(n) \leq \frac{8n\ell(\ell+1)}{(2\ell+1)^2-n}$$

(this is usually called relative bound) provided $n<(2\ell+1)^2$.

- For $k=2$ we have $P_4^{(n)}(t)=\frac{(n+2)(n+4)t^4-6(n+2)t^2+3}{n^2-1}$ and therefore

$$M_{2\ell+1}(n) \leq \frac{2(n-2)((2\ell+1)^4(n+2)+6(2\ell+1)^2-n-4)}{6(2\ell+1)^2(n+2)-3(2\ell+1)^4-(n+2)(n+4)} \quad (2)$$

provided $6(2\ell+1)^2(n+2)-3(2\ell+1)^4-(n+2)(n+4)>0$. The bound (2) is better than the relative bound for $n\geq 96$ and for every ℓ.

Generalizations?

- Free the inner products – consider codes with two possible inner products a and b (two-distance sets on $\mathbb{S}^{n-1}$).
  1. A. Barg, W-H. Yu, New upper bound for spherical two-distance sets, Experimental Math., 22, 2013, 187—194. arXiv:1204.5268
  2. A. Barg, A. Glazyrin, K. Okoudjou, W-H. Yu, Finite two-distance tight frames, Linear Algebra and its Application, 474, 2015, 163-175. arXiv:1402.3521
- Allow more inner products – this talk

Antipodal codes with inner products -1, 0 and $\pm s$ (1)

- $C \subset \mathbb{S}^{n-1}$ – antipodal, $M = |C|$, inner products -1, 0 and $\pm s$, where $0 < s < 1$.

Theorem

If $s^2 < \frac{3}{n+2}$, then

$$M \leq \frac{2n(n+2)(1-s^2)}{3-s^2(n+2)}. \quad (3)$$

Proof. Set $f(t) = t^2(t^2 - s^2)$ in (1). □

- If (3) is attained, then $M_2 = M_4 = 0$, i.e. C is a spherical 5-design. Then we compute the distance distribution $A_s(x) = A_s = \frac{M-2n}{2ns^2}$, $A_0(x) = A_0 = M - 2 - 2A_s = \frac{M(ns^2-1)+n(1-2s^2)}{ns^2}$

Antipodal codes with inner products -1, 0 and $\pm s$ (2)

We consider a derived code of C to obtain a Lloyd-type theorem.

Theorem

If C attains the bound (3) then s is rational.

Proof. Some algebraic manipulations. □

Theorem

If C is a spherical 3-design, $k \geq 2$ and $P_{2k}^{(n)}(s) + (ns^2 - 1)P_{2k}^{(n)}(0) < 0$, then

$$M \leq \frac{n\left(2ns + (1-2s^2)P_{2k}^{(n)}(0) - P_{2k}^{(n)}(s)\right)}{\left|P_{2k}^{(n)}(s) + (ns^2-1)P_{2k}^{(n)}(0)\right|}. \quad (4)$$

Proof. We set $f(t) = P_{2k}^{(n)}(t)$ in (1). □

Antipodal codes with inner products -1, $\pm s_1$ and $\pm s_2$ (1)

- $C \subset \mathbb{S}^{n-1}$ – antipodal, $M = |C|$, inner products -1, $\pm s_1$ and $\pm s_2$, where $0 < s_1 < s_2 < 1$. Again, we first derive the analog of the relative bound.

Theorem

If $s_1^2 s_2^2 + \frac{3-(n+2)(s_1^2+s_2^2)}{n(n+2)} > 0$ *and* $6-(n+4)(s_1^2+s_2^2) > 0$, *then*

$$M \le \frac{n(n+2)(1-s_1^2)(1-s_2^2)}{n(n+2)s_1^2 s_2^2 - (n+2)(s_1^2+s_2^2)+3}. \tag{5}$$

Proof. Set $f(t) = (t^2 - s_1^2)(t^2 - s_2^2)$ in (1). □

If (5) is attained, then C must be a spherical 5-design. Therefore

$$A_{s_1} = \frac{M-2n-ns_1^2(M-2)}{2n(s_1^2-s_2^2)}, \quad A_{s_2} = \frac{M-2n-ns_2^2(M-2)}{2n(s_1^2-s_2^2)}.$$

Antipodal codes with inner products -1, $\pm s_1$ and $\pm s_2$ (2)

- The investigation of the derived codes imply, similarly to the previous case, the following assertion.

Theorem

If C attains the bound (5) then s_1 are simultaneously rational or simultaneously irrational.

Proof. By calculation of the distance distribution of the derived codes $C_{s_1}(x)$ and $C_{s_2}(x)$. □

Antipodal codes with inner products -1, $\pm s_1$ and $\pm s_2$ (3)

- Analog of Theorems 1 and 4 follows from $M_{2k} \geq 0$.

Theorem

If C is a spherical 5-design, $k \geq 2$ and $(1-ns_1^2)P_{2k}^{(n)}(s_1)+(1-ns_2^2)P_{2k}^{(n)}(s_2)<0$, then

$$M \leq \frac{2n\left((1-s_1^2)P_{2k}^{(n)}(s_1)+(1-s_2^2)P_{2k}^{(n)}(s_2)+s_2^2-s_1^2\right)}{\left|(1-ns_1^2)P_{2k}^{(n)}(s_1)+(1-ns_2^2)P_{2k}^{(n)}(s_2)\right|}. \qquad (6)$$

Proof. Set $f(t)=P_{2k}^{(n)}(t)$ in (1). □

Thank you for your attention!