Colength of $*$-Polynomial Identities of Simple $*$-Algebras

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Introduction

- Let F be a field of characteristic 0;
- Let A be associative algebra over F;
- A function $* : A \to A$ is said to be an *involution* if $*$ is an automorphism of the additive group of A such that $(ab)^* = b^* a^*$ and $(a^*)^* = a$ for all $a, b \in A$.
- An example of such a map is the transpose in the algebra $M_n(F)$ of $n \times n$ matrices over the field F.

- Let $(A, *)$ be a unitary algebra with involution $*$;
- *Involution of the first kind*: the restriction of $*$ on F is the identical map
- *Involution of the second kind*: otherwise $*$ (the restriction of $*$ on F is $*$);
- The algebra $(A, *)$ is said to be $*$-*simple* if $A^2 \neq 0$ and it has no nontrivial $*$-invariant ideals.

- The *opposite algebra* of A, denoted by A^{op}, is the algebra that has the same elements as A, the same addition as A, and multiplication given by $a \circ b = ba$, where ba is a product in A.
- It is easy to check that $(A^{\mathrm{op}})^{\mathrm{op}} = A$, $A \cong B$ if and only if $A^{\mathrm{op}} \cong B^{\mathrm{op}}$.
- The algebra $A \oplus A^{\mathrm{op}}$ has the exchange involution defined by $(a, b)^* = (b, a)$.

Theorem (Rowen)

Let A be a $*$-simple finite dimensional associative algebra over an algebraically closed field. Then either A is simple as an algebra or A is of the form $A = B \oplus B^{\mathrm{op}}$, where B is a simple algebra.

- Let the field F be an algebraically closed field;
- Then the $*$-simple finite dimension algebras are:
  - $(M_n(F), t)$ with transpose;
  - $(M_n(F), s)$ symplectic involution (for even n);
  - $M_n(F) \oplus M_n(F)^{\mathrm{op}}$ with exchange involution.

- **Drensky and Giambruno** have obtained the exact values of the cocharacters, codimensions and the Hilbert series of the polynomial identities of the $*$-simple algebras $(M_2(F), t)$ and $(M_2(F), s)$.
- The subject of our study is the algebra $M_2(F) \oplus M_2(F)^{\text{op}}$ with the exchange involution.
- We obtain the sequence of colengths of its $*$-identities in the case when F is of characteristic zero.

- $(A, *)$ over field F, $\mathrm{char}(F) = 0$
- *The free associative algebra with involution* $F\langle X, *\rangle$ is the free associative algebra on the set of free generators $X \cup X^*$ where $X = \{x_1, x_2, \dots\}$ and $X^* = \{x_1^*, x_2^*, \dots\}$ and involution that extends the map $x_i \xrightarrow{*} x_i$ and $x_i^* \xrightarrow{*} x_i$.
- A polynomial $f(X, X^*) \in F\langle X, *\rangle$ is a $*$-*polynomial identity* for the algebra $(A, *)$ if $f(a_1, \dots, a_n; a_1^*, \dots, a_n^*) = 0$ for all $a_i \in A$.
- We denote by $T(A, *)$ the ideal of all $*$-polynomial identities of $(A, *)$.

- Let us denote the sets of symmetric and skew elements of A by $A^+ = \{a \in A \mid a^* = a\}$ and $A^- = \{a \in A \mid a^* = -a\}$, respectively.

- It is more convenient to change the variables x_i, x_i^* by $y_i = \frac{1}{2}(x_i + x_i^*),\ z_i = \frac{1}{2}(x_i - x_i^*)$ are the symmetric and skew variables, respectively.

- Then $F\langle X, *\rangle = F\langle Y, Z, *\rangle$ where $Y_p = \{y_1, \ldots, y_p\}$ is a set of symmetric variables $y_i \in F\langle Y, Z\rangle$, and $Z_q = \{z_1, \ldots, z_q\}$ is a set of skew variables $z_i \in F\langle Y, Z\rangle$.

- Consequently, $f(Y, Z) \in T(A, *)$ if and only if the polynomial $f(y_1, \ldots, y_p, z_1 \ldots, z_q)$ is such that $f(b_1, \ldots, b_p, c_1 \ldots, c_q) = 0$ for all $b_i \in A^+, i = 1, \ldots p$ and $c_j \in A^-,\ j = 1, \ldots, q$.

- The factor algebra $F(A,*) = F\langle Y, Z, *\rangle / T(A,*)$ is *the relatively free algebra in the variety of algebras with involution* generated by $(A,*)$.
- We denote by $F_{p,q}(A,*)$ the subalgebra of $F(A,*)$ generated by $Y_p = \{y_1, \ldots, y_p\}$ and $Z_q = \{z_1, \ldots z_q\}$ and assume that by $F_m(A,*) = F_{m,m}(A,*)$.

- The Hilbert series of $F_{p,q}(A,*)$ is defined as a formal power series

$$H(A,*,y_1,\dots,y_p,z_1\dots,z_q)=\sum_{(a,b)}\dim F_{p,q}^{(a,b)}y_1^{a_1}\dots y_p^{a_p}z_1^{b_1}\dots z_q^{b_q}$$

or if $Y_m^a=(y_1^{a_1}\dots y_p^{a_p})$ and $Z_m^b=(z_1^{b_1}\dots z_q^{b_q})$ then

$$H(A,*,Y_p,Z_q)=\sum_{(a,b)}\dim F_{p,q}^{(a,b)}Y_p^aZ_q^b$$

- For ordinary polynomial identities one of the most important numerical invariants of the polynomial identities of A is the S_n-cocharacter sequence

$$\chi_n(A) = \sum_{\lambda \vdash n} m_\lambda(A)\chi_\lambda, \quad n = 0, 1, 2, \dots,$$

where $\lambda = (\lambda_1, \dots, \lambda_n)$ is a partition of n and χ_λ is the corresponding irreducible character of the symmetric group S_n;

- The n-th cocharacter

$$\chi_n(A) = \chi_{S_n}(P_n/(T(A) \cap P_n))$$

is equal to the character of the representation of S_n acting on the vector subspace $P_n \subset K\langle X\rangle$ of the multilinear polynomials of degree n modulo the polynomial identities of A.

- For $*$-polynomial identities one considers the characters of the wreath product $\mathbb{Z}_2 \wr S_n$ where $\mathbb{Z}_2 = \{1, *\}$ is the multiplicative group of order 2, and S_n (**Giambruno and Regev**).
- The wreath product is defined by

$$\mathbb{Z}_2 \wr S_n = \{(a_1, \dots, a_n; \sigma) | a_i \in \mathbb{Z}_2, \sigma \in S_n\}$$

with multiplication given by

$$(a_1, \dots, a_n; \sigma)(b_1, \dots, b_n; \tau) = (a_1 b_{\sigma^{-1}(1)}, \dots, a_n b_{\sigma^{-1}(n)}; \sigma\tau).$$

- Let us denote by $\chi_{\lambda,\mu}$ the irreducible $\mathbb{Z}_2 \wr S_n$-character associated with the pair of partitions (λ, μ).

- The $\mathbb{Z}_2 \wr S_n$-module structure of the set of multilinear polynomilas in Y and Z (namely $P_n(A,*)$) and the $GL_m \times GL_m$-module structure of $F_m(A,*)$ are related by the following results given by **Giambruno**

Theorem

If

$$\chi_n(A,*) = \sum_{|\lambda|+|\mu|=n} m_{\lambda,\mu}\chi_{\lambda,\mu},$$

$$H(A,*,Y_m,Z_m) = \sum_{n\geq 0}\sum_{|\lambda|+|\mu|=n} b_{\lambda,\mu} S_\lambda(Y_m) S_\mu(Z_m),$$

then $m_{\lambda,\mu} = b_{\lambda,\mu}$ *for all* λ, μ, *where* $S_\lambda(Y_m)$ *and* $S_\mu(Z_m)$ *are the Schur functions indexed by* λ *and* μ, *respectively.*

Schur functions

$$S_\lambda(X) = \frac{V(\lambda+\delta, X)}{V(\lambda, X)},$$

where $\lambda = (\lambda_1, \dots, \lambda_d)$, $\delta = (d-1, d-2, \dots, 2, 1, 0)$ and $\mu = (\mu_1, \dots, \mu_d)$

$$V(\mu, X) = \begin{vmatrix} x_1^{\mu_1} & x_2^{\mu_1} & \dots & x_d^{\mu_1} \\ x_1^{\mu_2} & x_2^{\mu_2} & \dots & x_d^{\mu_2} \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{\mu_d} & x_2^{\mu_d} & \dots & x_d^{\mu_d} \end{vmatrix}$$

- In the $*$-case one considers the so-called Y-proper polynomial identities. They are the $*$-identities in which all symmetric variables participate in commutators only (**Drensky, Giambruno**).
- The Hilbert series of the relatively free algebra and its proper elements $(B_m(A,*))$ are related by

$$H(F_m(A,*), Y_p, Z_q) = H(B_m(A,*), Y_p, Z_q) \prod_{i=1}^{m} \frac{1}{1-y_i}.$$

- *Sequence of colengths* in the $*$-case is

$$l_n(A,*) = \sum_{k=0}^{n} l_{k,n-k}(A,*)$$

where

$$l_{k,n-k}(A,*) = \sum_{\substack{\lambda \vdash k \\ \mu \vdash n-k}} m_{\lambda\mu}, \ \ n = 1, 2, \ldots,$$

i.e. the sequence of lengths of the modules $P_n(A,*)$.

- **Drensky and Giambruno** have obtained the Hilbert series for the proper elements $B_{p,q}$ for algebra $M_2 \oplus M_2^{\mathrm{op}}$, i.e

$H(B_{p,q}, T_p, U_q) =$

$$\prod_{i=1}^{p} \frac{1}{1-t_i} \prod_{j=1}^{q} \frac{1}{(1-u_j)^2} \left(\sum_{n \geq 1} S_{(n,n)}(T_p, U_q) \right) - c(T_p, U_q),$$

$$\sum S_{(n,n)}(T_p, U_q) = \sum S_{(\lambda_1,\lambda_2)}(T_p) S_{(\mu_1,\mu_2)}(U_q),$$

where the summation runs on all (λ_1, λ_2) and (μ_1, μ_2) with $\lambda_1 + \mu_2 = \lambda_2 + \mu_1$

- and the correction $c(T_p, U_q)$ is

$$c(T_p, U_q) = \prod_{j=1}^{p} \frac{1}{1-u_j} \left(S_{(1^3)}(T_p, U_q) + \sum_{n \geq 1} S_{(n)}(T_p, U_q) \right).$$

- The description of the multiplicities $m_{\lambda,\mu}$ usually is given in terms of the multiplicity series of the polynomial $f(T_p, U_q)$

$$M(f, T_p, U_q) = \sum_{\lambda\mu} m_{\lambda\mu} T_p^{\lambda} U_q^{\mu}.$$

- Also we denote by $\mathcal{Y}_T$ (and similarly $\mathcal{Y}_U$) the **Young operator** which sends the multiplicities series of $f(T_p, U_q)$ to the multiplicities series of $\prod_{i=1}^{p} \frac{1}{1-t_i} f(T_p, U_q)$, i.e.,

$$\mathcal{Y}_T(M(f(T_p, U_q))) = M\left(\prod_{i=1}^{p} \frac{1}{1-t_i} f(T_p, U_q)\right)$$

- Using Young tableaux one can obtains the

$$H\left(\sum_{\lambda_1+\mu_2=\lambda_2+\mu_1} S_{(\lambda_1,\lambda_2)}(T_p)S_{(\mu_1,\mu_2)}(U_q)\right) =$$

$$\frac{1}{(1-t_1t_2)(1-u_1u_2)(1-t_1u_1)}$$

The main result of our paper is the following

Theorem

The colength series of the $*$-identities of $M_2 \oplus M_2^{op}$ is

$$l(t,u) = M(H(B(M_2 \oplus M_2^{op}), t, t, t, u, u, u, u)) = f_1 - f_2 - f_3,$$

where f_1, f_2 and f_3 are given bellow.

$$Nu := t^5u^9 + t^4u^8 - t^4u^4 + t^3u^7 - t^3u^4 - u^3t^3 - t^2u^6 - t^2u^5 + t^2u^2 - tu^5 + tu + 1$$
$$De := (1+u^2)(1-t)^2(1-t^3)(1-t^2u)(1+u)^3(1+t)(1-tu)^2(1+tu)^2(1-tu^2)^2 \times$$
$$\times (1-u^3)^2(1-u)^5$$

$$f1 = \frac{Nu}{De}$$

$$f_2 = \prod_{j=1}^{p} \frac{1}{1-u_j} S_{(1,1,1)}(T_p, U_q) = \prod_{j=1}^{p} \frac{1}{1-u_j} \sum_{k=1}^{3} S_{(1^k)}(T_p) S_{(1^{3-k})}(U_q) = \prod_{j=1}^{p} \frac{1}{1-u_j} \times$$

$$\times \Big(S_{(0)}(T_3) S_{(1^3)}(U_4) + S_{(1)}(T_3) S_{(1^2)}(U_4) + S_{(1^2)}(T_3) S_{(1)}(U_4) + S_{(1^3)}(T_3) S_{(0)}(U_4) \Big)$$

$$= \frac{u^3 + tu^2 + t^2 u + t^3}{(1-u)^4}$$

$$f_3 = \prod_{j=1}^{p} \frac{1}{1-u_j} S_{(n)}(T_p, U_q) = \prod_{j=1}^{p} \frac{1}{1-u_j} \sum_{n \geq 1} S_{(k)} \times S_{(n-k)}$$

$$= \frac{1}{(1-u)^4} \left(\frac{1}{(1-t)^3 (1-u)^4} - 1 \right)$$

- $M(f, T, U) = \sum_n \sum_k \sum_{\lambda,\mu} m_{\lambda\mu} T^\lambda U^\mu$
- When one substitutes $t_i = t, i = 1, \ldots, p$ and $u_j = u$ for $j = 1, \ldots, q$ then obtains

$$M(f, t, u) = \sum_n \sum_k \left(\sum_{\lambda,\mu} m_{\lambda\mu} \right) t^k u^{n-k} = \sum_n \sum_k l_{k,n-k} t^k u^{n-k}.$$

- The colength $l_{k,n-k} = \sum_{\lambda,\mu} m_{\lambda\mu}$.

THANK YOU!