New Polynomials for Strong Algebraic Manipulation Detection Codes

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XV Int. Workshop on Algebraic and Combinatorial Coding Theory
(ACCT2016)
Albena, Bulgaria

June 23, 2016

Overview

Algebraic Manipulations

An algebraic manipulation is a model of an undesirable data modification [Jongsma'08].
Briefly, an additive data distortion is called an algebraic manipulation if its value does not depend on a value-to-be-distorted.

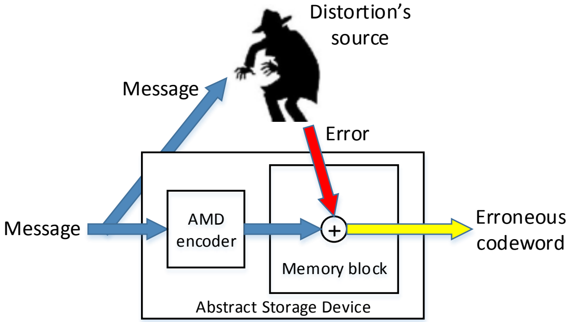
Algebraic manipulation detection (AMD) codes are designed to detect algebraic manipulations.

Types of AMD codes:

- weak AMD codes (also known as robust codes)
- strong AMD codes
    - include "stronger" AMD codes

Only systematic strong AMD codes over $GF(2^n)$ will be examined.

Strong Algebraic Manipulations



A strong algebraic manipulation model assumes:

- An additive error
- An error may be any nonzero element of $GF(2^n)$
- There is a dependency between an error and a message
- The Hamming metric

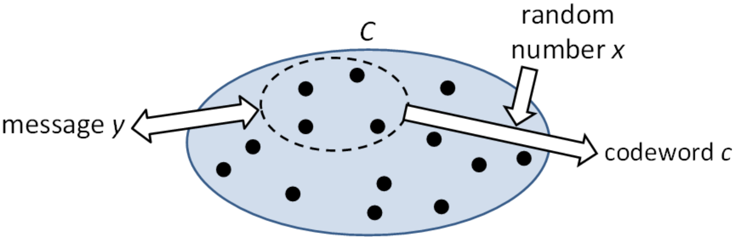
Code Construction

A codeword

$$c = (y, \quad x, \quad f(x,y))$$

consists of 3 parts:

- an informational message $y \in GF(2^k)$,
- a random number $x \in GF(2^m)$,
- a check symbol $f(x,y) \in GF(2^r)$.



Decoding

For a distorted codeword

$$\begin{aligned}\tilde{c} = c + e &= (y + e_y, x + e_x, f(x, y) + e_f) \\ &= (\tilde{y}, \qquad \tilde{x}, \qquad \tilde{f}(x, y)).\end{aligned}$$

compute a syndrome:

$$S(\tilde{c}) = f(\tilde{x}, \tilde{y}) + \tilde{f}(x, y).$$

If $S(\tilde{c}) = 0 \in GF(2^r)$, then no errors detected,
otherwise an error is detected.

The main advantage of AMD codes comparing to classic linear codes is that every q-ary linear code has $q^k - 1$ nonzero undetectable errors, where k is a dimension of a code.
AMD codes have no undetectable errors, each error is detected with some nonzero probability $1 - P_{undet}$.
P_{undet} is a worst–case probability of error masking (an achievable bound).

Applications

Were introduced in 2008 to detect cheaters in linear secret sharing schemes [Cramer et al.'08].

More applications have been found [Cramer et al.'13]:

- Design of secure cryptographic devices (fault-injection attack, ...);
- Fault–tolerant storage devices;
- Robust fuzzy extractors;
- Non-malleable codes;
- Anonymous quantum communication; and others.

Definition of Strong AMD Codes

Definition 1 (Strong AMD Codes)

A code

$$C = \{(y, x, f(x, y))\}$$

is a systematic strong AMD code if the encoding function $f(x, y)$ *satisfies the following inequality:*

$$P_{undet} \leq \max_{y, e: e_y \neq 0} \frac{|\{x : S(\tilde{c}) = 0\}|}{|\{x\}|} < 1. \tag{1}$$

Classic strong AMD codes are based on [Cramer et al.'13]:

- Message authentication codes
- Error correction codes
- A multiplication in a finite field, and others.

Definition of Stronger AMD Codes

There is a subset of strong AMD codes that are called *stronger AMD codes*.

Definition 2 (Stronger AMD Codes)

Stronger codes satisfy the equation (1) for all $e \neq 0 \in GF(2^n)$, not only for $e : e_y \neq 0$:

$$P_{undet} \leq \max_{y, e \neq 0} \frac{|\{x : S(\tilde{c}) = 0\}|}{|\{x\}|} < 1. \qquad (2)$$

There are two families of stronger AMD codes [Alekseev'15].
The most efficient one is based on polynomials. Initially, the next encoding polynomial was proposed [Cramer et al.'08]:

$$f(x, y) = y_1 x + y_2 x^2 + \ldots + y_t x^t + x^{t+2}.$$

Examples of Stronger AMD Codes

Karpovsky et al. developed this code into a sophisticated and flexible construction with a variety of encoding polynomials for different parameters [Karpovsky et al.'14].
An encoding function is always a sum of two polynomials:

$$f(x,y) = A(x) + B(y,x).$$

Another example of a stronger AMD code [Karpovsky et al.'14]

The code with $r = 2$ bits,
$k = 4r$ bits $(y \to (y_1, y_2, y_3, y_4))$,
$m = 2r$ (two variables $x \to (x_1, x_2)$),
$x_i, y_i \in GF(2^r)$
has the following encoding polynomial:

$$f(x,y) = A(x) + B(y,x) = \left(x_1 x_2^3 + x_1^3 x_2\right) + \left(y_1 x_1 + y_2 x_1^2 + y_3 x_2 + y_4 x_2^2\right).$$

Proposed family of polynomials

Let $y \in GF(2^{k=ar})$, $y \to (y_1, y_2, \dots, y_a)$, $y_i \in GF(2^r)$, and $x \in GF(2^{m=br})$, $x \to (x_1, x_2, \dots, x_b)$, $x_j \in GF(2^r)$, $a, b, r \geq 1$.
Let us define the following family of polynomial functions:

$$\begin{aligned} f(x,y) &= y_1 x_1^{\alpha_{1,1}} \dots x_b^{\alpha_{1,b}} + y_2 x_1^{\alpha_{2,1}} \dots x_b^{\alpha_{2,b}} + \dots + y_a x_1^{\alpha_{a,1}} \dots x_b^{\alpha_{a,b}} \\ &= \sum_{i=1}^{a} y_i x_1^{\alpha_{i,1}} x_2^{\alpha_{i,2}} \dots x_b^{\alpha_{i,b}} = \sum_{i=1}^{a} y_i \prod_{j=1}^{b} x_j^{\alpha_{i,j}}, \end{aligned} \tag{3}$$

where $\alpha_{i,j} \in \{0, 2^l\}$, $0 \leq l < r$.

For each consecutive i, a new set of $\alpha_{i,j}$ is selected in order to minimize the sum $\sum_j \alpha_{i,j}$, and the set of all zeros is prohibited.
A number of available sets of α is limited due to the restriction:

$$\sum_j \alpha_{i,j} < r.$$

Examples of proposed polynomials - I

An example polynomial #1

Let $a = 2$, $b = 1$, thus, $y \to (y_1, y_2)$ and there is one variable x. Then the next sets of α are chosen:

$$\begin{aligned} y_1: &\qquad \alpha_{1,1} = 2^0 &\qquad \to x^1, \\ y_2: &\qquad \alpha_{2,1} = 2^1 &\qquad \to x^2. \end{aligned}$$

The obtained polynomial is:

$$f(x, y) = y_1 x^{2^0} + y_2 x^{2^1} = y_1 x + y_2 x^2.$$

Examples of proposed polynomials - II

An example polynomial #2

Let $a = 3$, $b = 3$, thus, $y \to (y_1, y_2, y_3)$ and $x \to (x_1, x_2, x_3)$. The next sets of α are chosen:

$$
\begin{array}{lll}
y_1: & \alpha_{1,1} = 2^0, \alpha_{1,2} = 0, \alpha_{1,3} = 0 & \to x_1^1 x_2^0 x_3^0 = x_1, \\
y_2: & \alpha_{2,1} = 0, \alpha_{2,2} = 2^0, \alpha_{2,3} = 0 & \to x_1^0 x_2^1 x_3^0 = x_2, \\
y_3: & \alpha_{3,1} = 0, \alpha_{3,2} = 0, \alpha_{3,3} = 2^0 & \to x_1^0 x_2^0 x_3^1 = x_3.
\end{array}
$$

The following polynomial is constructed:

$$f(x, y) = y_1 x_1 + y_2 x_2 + y_3 x_3.$$

Examples of proposed polynomials - III

An example polynomial #3

Let $a = 6$, $b = 2$, therefore, $y \to (y_1, \ldots, y_6)$ and $x \to (x_1, x_2)$. Then the next sets of α are chosen:

$$
\begin{aligned}
y_1 : \quad & \alpha_{1,1} = 2^0, \alpha_{1,2} = 0 && \to x_1^1 x_2^0 = x_1, \\
y_2 : \quad & \alpha_{2,1} = 0, \alpha_{2,2} = 2^0 && \to x_1^0 x_2^1 = x_2, \\
y_3 : \quad & \alpha_{3,1} = 2^1, \alpha_{3,2} = 0 && \to x_1^2 x_2^0 = x_1^2, \\
y_4 : \quad & \alpha_{4,1} = 0, \alpha_{4,2} = 2^1 && \to x_1^0 x_2^2 = x_2^2, \\
y_5 : \quad & \alpha_{5,1} = 2^0, \alpha_{5,2} = 2^0 && \to x_1^1 x_2^1 = x_1 x_2, \\
y_6 : \quad & \alpha_{6,1} = 2^0, \alpha_{6,2} = 2^1 && \to x_1^1 x_2^2 = x_1 x_2^2.
\end{aligned}
$$

The constructed polynomial is:

$$f(x, y) = y_1 x_1 + y_2 x_2 + y_3 x_1^2 + y_4 x_2^2 + y_5 x_1 x_2 + y_6 x_1 x_2^2.$$

Code construction

Theorem

$$C = \{\big(y \in GF(2^{ar}), x \in GF(2^{br}), f(x,y) \in GF(2^r)\big)\}$$

with an encoding function $f(x,y)$ defined by the equation (3) is a strong AMD code providing an error masking probability

$$P_{undet} \leq 1 - (2^r - v)2^{-(u+1)r},$$

where p is the power of the encoding polynomial, and $p = u(2^r - 1) + v$, $u \leq b$, $v < 2^r - 1$.

The proof is based on the following property of a Galois field $GF(p^m)$:

$$(a+b)^{p^i} = a^{p^i} + b^{p^i}.$$

A code construction defined by the Theorem has the same formula of an error masking probability as stronger codes based on polynomials [Karpovsky et al.'14].

Performance comparison

Let p be a power of a polynomial for a stronger code from [Karpovsky et al.'14] with parameters k, m and r, and p' be a power of a proposed polynomial for same parameters. Then:

- if $p < p - 1$, a proposed code provides lower P_{undet} and lower computational complexity.
- If $p' = p - 1$, a proposed code provides the same P_{undet}, but its polynomial has a lower power and less monomials (thus, lower complexity).
- if $p' \geq p$, a stronger code is more efficient than a proposed one.

A replacement of stronger AMD codes with proposed strong ones is feasible only in cases when it is sufficient to provide error detection in informational parts y of codewords (not in all parts).
However, this requirement seems to be adequate for most applications.

Some cases when proposed codes are better - I

In general, a power of a proposed polynomial grows faster than that of stronger codes.
However, for small $a = k/r$ it is possible to construct a strong code with a lower power of a polynomial.

Example 1:

$k = 8$ bits i.e. $y \in GF(2^8)$,
$m = 4$ bits i.e. $x \in GF(2^4)$,
$r = 4$ bits i.e. $f(x, y) \in GF(2^4)$.

Therefore, $a = k/r = 2$, $y \to (y_1, y_2)$,
and $b = m/r = 1$, $x \to x$,
$y_i, x_i \in GF(2^4)$.

Code	$f(x, y)$	P_{undet}
Stronger	$y_1 x + y_2 x^2 + x^5$	0.25
Proposed	$y_1 x + y_2 x^2$	0.125

Some cases when proposed codes are better - II

Example 2:
$k = 12$ bits i.e. $y \in GF(2^{12})$,
$m = 12$ bits i.e. $x \in GF(2^{12})$,
$r = 4$ bits i.e. $f(x, y) \in GF(2^4)$.

Therefore, $a = k/r = 3$, $y \to (y_1, y_2, y_3)$,
and $b = m/r = 3$, $x \to (x_1, x_2, x_3)$,
$y_i, x_i \in GF(2^4)$.

Code	$f(x, y)$	P_{undet}
Stonger	$y_1x_1 + y_2x_2 + y_3x_3 + x_1^3 + x_2^3 + x_3^3$	0.125
Proposed	$y_1x_1 + y_2x_2 + y_3x_3$	0.06

Some cases when proposed codes are better - III

Example 3:
$k = 24$ bits i.e. $y \in GF(2^{24})$,
$m = 8$ bits i.e. $x \in GF(2^8)$,
$r = 4$ bits i.e. $f(x, y) \in GF(2^4)$.

Therefore, $a = k/r = 6$, $y \to (y_1, \ldots, y_6)$,
and $b = m/r = 2$, $x \to (x_1, x_2)$,
$y_i, x_i \in GF(2^4)$.

Code	$f(x,y)$	P_{undet}
Stronger	$y_1x_1 + y_2x_2 + y_3x_1^2 + y_4x_2^2 + y_5x_1x_2 + y_6x_1^3 + x_1x_2^3$	0.188
Proposed	$y_1x_1 + y_2x_2 + y_3x_1^2 + y_4x_2^2 + y_5x_1x_2 + y_6x_1x_2^2$	0.188

Summary:

- A new family of polynomial encoding functions of strong AMD codes is presented.
- In some cases proposed ones have less monomials (in fact, a part $A(x)$ of $f(x,y)$ is omitted) and a lower power. This leads to a lower error masking probability and lower computational complexity.
- Efficient encoding and decoding methods based on the Horner scheme can be used for proposed codes [Karpovsky et al.'14].

To do:

- Find a formula to compute a power of $f(x,y)$ (and, thus, P_{undet}) for each set of k, m and r (determine when proposed codes are better).
- Check if codes lay on bounds.

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Thank you for your attention!