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Search for a moving target in a graph

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Abstract. In this paper we consider a searching game called k-chase. A Princess occupies a vertex of a given graph G and a Suitor is trying to find her. On each turn, the Suitor examines k vertices of G looking for the Princess (and, if he finds her, the game ends). Following this, the Princess moves to an adjacent vertex of G and the turn is complete. For $k = 1$, we give a complete characterization of graphs for which it is possible for the Suitor to find the Princess. We also find the minimum k for which the Suitor finds the Princess when G is a rectangular grid of size $2n \times 2n$.

1 Preliminaries

In general, search problems can be viewed as a game between two players, a *Questioner* and a *Responder*. The game is played as follows: first, the Responder selects an element x, unknown to the Questioner, out of some fixed set S (the *search space*). The Questioner then attempts to find x by asking questions of a pre-specified form, e.g., whether x is in a given subset of S. For more information and extensive bibliography the reader is referred to [1].

In this paper we study a searching game in which the unknown element moves within the search space. Searching games of this kind have previously been considered in [2].

Definition. Let G be a finite directed graph. A *k–chase* on G is a game played between two players, a *Suitor* and a *Princess*. At the beginning of the game, the Princess occupies a vertex of G unknown to the Suitor. On each turn the Suitor investigates k vertices of G. If the Princess occupies one of them, the Suitor finds her, wins and the game ends. Otherwise, the Princess moves to an adjacent vertex and the turn is complete. If the Princess can evade the Suitor indefinitely, the game is a win for the Princess.

This differs from classical moving target search on two counts: the upper bound on question size (ordinarily, questioning arbitrary subsets of the search space is allowed) and vertices being investigated individually rather than as a set (i.e., it suffices for the Suitor to name just one subset of the search space containing the Princess).

Let $S \subset V(G)$. Denote by $c(S)$ the set of all direct successors of the vertices in S. Given a strategy for the Suitor, we write S_i for the set of vertices that the Princess could occupy at the beginning of turn i and Q_i for the set of vertices investigated by the Suitor on turn i.

So, $S_1 = V(G)$ and for all i we have $S_{i+1} = c(S_i \setminus Q_i)$. Moreover, a k-chase is a win for the Suitor if and only if for some i he can achieve $|S_i| \leq k$.

Definition. The *chase depth* of a given graph G is the minimum positive integer k such that the Suitor has a winning strategy for the k-chase on G.

In the given scenario there are three main research questions:

Q1. Given a positive integer k, describe all graphs G of chase depth k.

Q2. Given a graph G, find its chase depth.

Q3. Given a graph G of chase depth k, find the minimum number of turns necessary for the Suitor to win.

Very little has been written on the above problems. A special case of Q3 (when G is a path of length 17 and $k = 1$) was suggested by Dave Penneys [3], initiating the line of research that led to the results presented in this paper. We focus on Q1 for $k = 1$ and on Q2 for rectangular grid graphs. We begin by listing some general properties aimed toward establishing our main results.

2 General results

First we study the connection between a connected graph and its chase depth.

Definition. The *stratification* of a connected graph G is the finest partitioning $O_1, O_2, \ldots, O_m$ of $V(G)$ such that $c(O_i) = O_{i+1}$ for all i (where $O_{m+1} = O_1$). The O_i are the *strata* of G.

The stratification of a connected graph G can be obtained as follows. Start with $O'_1 = \{v\}$ where v is any vertex of G, and set $O'_{i+1} = c(O'_i)$ for all i. The sequence $O'_1, O'_2, \ldots$ is eventually periodic and its period is exactly the stratification of G.

Lemma 1. *Stratification lemma.* The chase depth of a graph G with stratification $O_1, O_2, \ldots, O_m$ does not change if, at the beginning of the game, the Suitor is given the additional information that the Princess occupies a vertex in O_1.

Proof. Let k' be the O_1-chase depth of G, i.e., the least positive integer such that there exists a winning strategy for the Suitor given that the Princess occupies a vertex in O_1.

Clearly, $k \geq k'$. We are left to exhibit a winning k'-depth chase strategy for the Suitor.

The Suitor proceeds as follows. On stage 1, he carries out the O_1-chase strategy. If this does not result in the discovery of the Princess, the Suitor plays waiting moves until some turn i_1 such that $i_1 + 1$ is divisible by m and, on stage 2, carries out the O_1-chase strategy once again. If this does not result in the discovery of the Princess, the Suitor plays waiting moves until some further turn i_2 such that $i_2 + 2$ is divisible by m, et cetera. This guarantees that, if the Princess starts in O_j, the Suitor finds her on stage j. □

Notice that the stratification lemma does not tell us much about the number of turns that it takes for the Suitor to find the Princess relative to the case of an O_1-chase.

Lemma 2. *Expansion lemma.* Let G be a graph with stratification O_1, O_2, $\ldots$, O_m and chase depth k. Suppose that there exist positive integers l, l_1, l_2, $\ldots$, l_m such that, for all i, $l_i < |O_i|$ and every subset S of O_i of size at least l_i satisfies $|c(S)| \geq l + l_{i+1}$ (where $l_{m+1} = l_1$). Then $k \geq l + 1$.

Proof. Suppose that $k \leq l$ and let the Princess start at O_j. Then $S_1 = O_j$ and the condition of the Lemma implies that $|S_t| \geq l + l_{t+j-1}$ (where $l_{m+1} = l_1$). Therefore, $|S_t| \geq l_t + l > l \geq k$ for all t, implying that $|S_t| > k$ for all t. Hence, the Princess wins. □

From this point on, we shall only be interested in symmetric, loopless graphs, or, equivalently, in undirected graphs.

Lemma 3. Let $O_1, O_2, \ldots, O_m$ be the stratification of an undirected graph G. Then $m = 2$ if G is bipartite and $m = 1$ otherwise.

Proof. Since G is undirected, $O_1 \subset c(c(O_1)) = O_3$, so $m \leq 2$. □

Even though we make heavy use of the expansion lemma in subsequent exposition, it has its limitations even in the undirected case.

Theorem 1. There exist undirected graphs G of arbitrarily large depth k such that the best bound obtained by means of the expansion lemma is $\frac{k+1}{2}$.

Proof. Construct G as follows. Start with a complete graph on $2r + 1$ red vertices. For each pair u, v of red vertices, add $2r + 1$ green vertices adjacent only to u and v, or $g = (2r + 1)\binom{2r+1}{2}$ green vertices total.

The stratification of G consists of a single stratum, $O_1 = V(G)$. We prove first that for each positive integer $l_1 < |O_1|$, there is a subset S of $V(G)$ such that $|S| = l_1$ and $|c(S)| \leq l_1 + r$. Indeed,

- if $l_1 \leq r$, any set of l_1 green vertices has at most $2l_1$ neighbours, thus $|c(S)| \leq 2l_1 \leq l_1 + r$;
- if $r + 1 \leq l_1 \leq g$, any set of green vertices has at most $2r + 1$ neighbours, thus $|c(S)| \leq 2r + 1 \leq l_1 + r$;
- if $g < l_1 < |V(G)| - r = g + r + 1$, the set of all green vertices together with $l_1 - g \leq r$ red vertices has at most $(2r + 1) + [g - (2r + 1)] = g$ neighbours, thus $|c(S)| \leq g < l_1 + r$;
- if $l_1 \geq |V(G)| - r$ the statement is obvious.

The Expansion Lemma implies that we cannot have a depth bound greater than $r + 1$.

On the other hand, no $2r$-depth strategy exists for the Suitor as, when S_1 is the set of all red vertices, S_i would contain at least one red vertex for all odd i and at least one green vertex for all even i. □

3 Graphs of chase width 1

In this section we describe all simple graphs G of chase depth 1.

Definition. A *path* of length n is a graph with vertex set $\{v_1, v_2, \dots, v_n\}$ and edges $v_i v_{i+1}$ for $1 \le i \le n-1$.

Lemma 4. Any path has chase depth 1.

Proof. Such a graph is bipartite with strata $O_1 = \{v_1, v_3, \dots\}$ and $O_2 = \{v_2, v_4, \dots\}$. By the stratification lemma, we may assume that the Princess is in O_2. The Suitor finds her by successively looking into vertices v_2, v_4, $\dots$, v_{n-1}. □

Remark. For a path of length n and $k = 1$, the minimum number of turns necessary for the Suitor to win is $2n - 2$.

Lemma 5. Any cycle has chase depth 2.

Proof. Suppose that the cycle $C_n = (v_1, v_2, \dots, v_n)$ has chase depth 1 and consider a winning strategy for the Suitor such that he examines vertex u_j on turn j. Since the Princess has two options on every turn, she can always play so that she does not occupy u_j on turn j: a contradiction.

On the other hand, for $k = 2$ the Suitor can examine v_1 on each turn and, for the second vertex, simulate the path strategy specified in the proof of Lemma 4. □

Lemma 6. The graph $G^\star$ with vertices $\{u_i \mid 0 \le i \le 9\}$ and edges $\{(u_0, u_i) \mid i = 1,2,3\} \cup \{(u_{3k+i-3}, u_{3k+i}) \mid k = 1,2; i = 1,2,3\}$ has chase depth 2.

Proof. Apply the expansion lemma for $O_1 = \{u_0, u_4, u_5, u_6\}$, $O_2 = V(G) \backslash A$, $l = 1$, $l_1 = 2$ and $l_2 = 3$. □

The following Theorem describes all graphs of chase depth 1.

Theorem 2. A graph G has chase depth 1 if and only if G is acyclic and $G^\star \not\subset G$.

Proof. If G has a cycle or $G^\star \subset G$ then Lemma 5 and Lemma 6 imply that the chase depth of G is at least 2.

If G is a path then the result follows by Lemma 4. Suppose, from this point on, that G is not a path.

Let $v_1 v_2 \dots v_n$ be the longest path in G. Since G is acyclic and $G^\star \not\subset G$, we infer that for any i the longest path from v_i to a vertex outside $v_1, v_2, \dots, v_n$ has maximal length:

- 0 for $i = 1$ and $i = n$;
- 1 for $i = 2$ and $i = n - 1$;
- 2 for $i = 3, \dots, n - 2$.

Since G is acyclic, G is bipartite. Hence G has two strata, say, O_1 and O_2 with $v_2 \in O_2$. By the stratification lemma, we may assume that the Princess is initially in O_2. Consider the following strategy.

The Suitor begins by successively examining v_2, v_3, ... until he reaches a vertex v_j with $3 \leq j \leq n-2$ of degree at least 3. Let the adjacent vertices of v_j distinct from v_{j-1} and v_{j+1} be u_1, u_2, ..., u_k. In the worst-case scenario, there exist vertices w_1, w_2, ..., w_k, each of degree 1, such that u_i and w_i are adjacent for $1 \leq i \leq k$. Note also that v_j and w_1, w_2, ..., w_k lie in one and the same stratum. Prior to examining v_j, the Suitor knows that the Princess occupies the stratum containing v_j. Thus, if the Princess occupies vertex w then one of the following is true:

- $w = v_s$ for $s \geq j$ and $j + s$ even;
- w is at a distance of 2 from v_s for $s > j$ and $j + s$ even;
- w is adjacent to v_s for $s > j$ and $j + s$ odd;
- $w = w_s$ for some $1 \leq s \leq k$.

Denote the set of all possible w by S. After the Suitor looks into v_j and u_1, the Princess could only occupy a vertex in $S \backslash \{w_1\}$. The Suitor looks into v_j and u_2 next: after this, the Princess could only occupy a vertex in $S \backslash \{w_1, w_2\}$. Continuing on in this way, after the sequence v_j, u_1, v_j, u_2, ..., v_j, u_k, the Princess occupies a vertex in

$$S \backslash \{w_1, w_2, \ldots, w_k\}.$$

The Suitor continues by examining v_{j+1} and applying the same strategy recursively. Therefore, G has chase depth 1. □

4 Rectangular grids

In this section we find the chase depth of a rectangular grid of size $2n \times 2n$.

Definition. An $m \times n$ *grid* is an undirected graph of vertices (i, j), $1 \leq i \leq m$ and $1 \leq j \leq n$, such that (i_1, j_1) and (i_2, j_2) are adjacent iff

$$\{|i_1 - i_2|, |j_1 - j_2|\} = \{0, 1\}.$$

An $m \times n$ grid is isomorphic to an $m \times n$ chessboard with two squares being adjacent iff they share a common side.

Theorem 3. The chase depth of a $2n \times 2n$ grid is $n + 1$.

Proof. Take O_1 to be the set of all white squares ($i + j$ even) and O_2 to be the set of all black squares ($i + j$ odd).

First we exhibit a winning $(n+1)$-width O_1-chase strategy for the Suitor. This is exceedingly simple: number all squares 1 through $4n^2$ from left to right and from bottom to top (so that (i, j) is numbered $n(i-1) + j$) and then, on each turn i, investigate the $n + 1$ lowest-ranking squares in S_i. It is easy to see that this leads to $|S_{i+1}| = |S_i| - 1$ for all i until the Princess is found.

We proceed to show that every set S of n^2 white squares satisfies $|c(S)| \geq n^2 + n$. By the expansion lemma with $l = n$ and $l_1 = l_2 = n^2$, this suffices to complete the proof.

Partition $V(G)$ into n vertical $2 \times n$ rectangles B_p, $1 \leq p \leq n$ (so that $B_p = \{(i,j) \mid 2p-1 \leq i \leq 2p\}$).

We say that a B_p is *homogenous* iff either all white squares in B_p do not belong to S or all white squares in B_p belong to S.

If B_p is non-homogenous, then $|c(S) \cap B_p| \geq |S \cap B_p| + 1$. It follows that, if all B_p are non-homogenous, $|c(S)| \geq |S| + n$, and we are done.

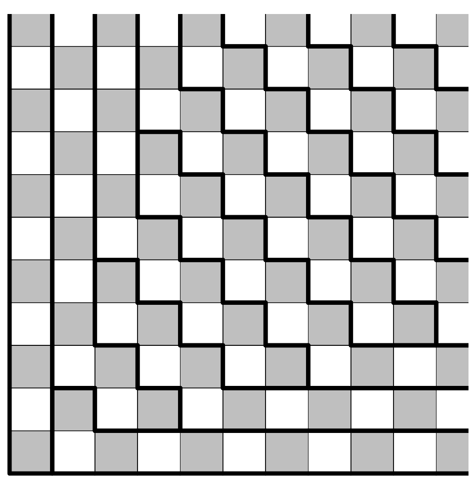
We are left to consider the case when at least one B_p is homogenous.

Repeat the above argument, but with $V(G)$ partitioned into horizontal rectangles C_q, $1 \leq q \leq n$ (so that $C_q = \{(i,j) \mid 2q-1 \leq j \leq 2q\}$). We see that there is at least one homogenous C_q.

Since B_p and C_q have (exactly two) common white squares, they must be homogenous in the same way. This means that either all white squares in $B_p \cup C_q$ are not in S, or all white squares in $B_p \cup C_q$ are in S. Assume the former; the latter case is analogous.

Lemma 7. Consider a $(2a+1) \times (2b+1)$ grid H, $a \geq b$. Colour the squares of H in a checker-like manner (so that the colour of (i,j) is a bijective function of the parity of $i+j$) and remove all white squares in the top row and the rightmost column. Then the remainder of H can be partitioned into disjoint paths so that each path starts and ends on a black square and the length of each path is at most $2a+5$.

Proof. The figure exhibits the lower left corner of a partitioning I of the upper right quadrant into disjoint paths (it is clear from the figure how I extends to the complete quadrant). Given a and b, restrict I to the rectangle spanned by either $(1,1)$ and (a,b), or $(2,1)$ and $(a+1,b)$, depending on H's colouring, and remove all white squares in the top row and rightmost column of the restriction. This gives the desired partitioning; moreover, with this construction the lengths of all paths do not exceed $2a+3$. □



Partition G into four disjoint rectangles along the longer axes of B_p and C_q, then partition each quarter as in the lemma. Since each of the paths obtained in this way contains at most $n+1$ white squares and S consists of n^2 white squares, there are n paths $P_1, P_2, \ldots, P_n$ such that $P_i \cap S$ is non-empty for all i. This, however, means that $|c(S) \cap P_i| \geq |S \cap P_i| + 1$ for all i and, consequently, $|c(S)| \geq |S| + n$, and we are done. $\square$

References

[1] Avieri S. Fraenkel, Combinatorial Games: selected bibliography with a succinct gourmet introduction, *The electronic journal of combinatorics*, DS2, 2009.

[2] Harout Aydinian, Ferdinando Cicalese , Christian Deppe, Vladimir Lebedev, A Combinatorial Model of Two-Sided Search, *SOFSEM 2016: Theory and Practice of Computer Science*, Volume 9587, Lecture Notes in Computer Science, 148-160, 2016.

[3] Gary Antonick, The Princess Problem, *Wordplay. The Crossword Blog of the New York Times*, 2014, `http://wordplay.blogs.nytimes.com/2014/01/27/princess/`.