Fifteenth International Workshop on Algebraic and Combinatorial Coding Theory
June 18-24, 2016, Albena, Bulgaria *pp. 267–271*

Characterization of Highly Divisible Arcs [1]

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Abstract. We prove that if the restriction of a $(t \mod q)$-arc $\mathcal{K}$ to every hyperplane is a lifted arc then $\mathcal{K}$ is itself lifted. We use this result to prove that all $(2 \mod q)$-arcs in $\mathrm{PG}(r,q)$, $r \geq 3$, are lifted.

1 Introduction

Let $\mathcal{K}$ be an (n,w)-arc with spectrum (a_i) It is said to be divisible with divisor $\Delta > 1$ if $a_i = 0$ for all $i \not\equiv n \pmod{\Delta}$. If the arc $\mathcal{K}$ satisfies $w \equiv n+t \pmod{q}$ and $a_i = 0$ for all $i \not\equiv n, n+1, \ldots, n+t \pmod{\Delta}$, $1 \leq t \leq q-1$, we say that it is t-quasidivisible with divisor $\Delta > 1$ (or t-quasidivisible modulo Δ). Let t be a fixed non-negative integer. An arc $\mathcal{K}$ in Σ is called a $(t \mod q)$-arc if

(1) for every point $P \in \mathcal{P}$, $\mathcal{K}(P) \leq t$;

(2) for every subspace S of dimension at least 1, $\mathcal{K}(S) \equiv t \pmod{q}$.

It turns out that $(t \mod q)$-arcs arise naturally as duals of t-quasidivisible arcs. Let $\mathcal{K}$ be a t-quasidivisible (n,w)-arc with divisor q in Σ, $t < q$. Denote by $\widetilde{\mathcal{K}}$ the arc

$$\widetilde{\mathcal{K}} : \begin{cases} \mathcal{H} & \rightarrow \quad \{0,1,\ldots,t\} \\ H & \rightarrow \quad \widetilde{\mathcal{K}}(H) \equiv n+t-\mathcal{K}(H) \pmod{q} \end{cases}, \tag{1}$$

where $\mathcal{H}$ is the set of all hyperplanes in Σ. This means that hyperplanes of multiplicity congruent to $n+a \pmod{q}$ become $(t-a)$-points in the dual geometry. I Then $\widetilde{\mathcal{K}}$ is a $(t \mod q)$-arc [5, 8]. For a more detailed introduction to arcs and blocking sets and their relation to linear codes, we refer to [1, 4].

The aim of this talk is to present various constructions and structure results for $(t \mod q)$-arcs. Section 2 contains general constructions for $(t \mod q)$-arcs. The most important is the so-called lifting construction, which is partly due to the fact that in dimension higher than 3 the only known $(t \mod q)$-arcs are sums of lifted arcs. In section 3, we prove that every $(2 \mod q)$-arc is lifted.

[1]The authors gratefully acknowledge the financial support by the Scientific Research Fund of Sofia University under Contract 29/11.04.2016.

This result implies Maruta's extendability result for linear codes with weights $-2,-1,0 \mod q$ for q odd. In section 4, we characterize the $(3 \mod 5)$-arcs of small cardinality and prove that every $(3 \mod 5)$-arc in $\mathrm{PG}(3,5)$ of size not exceeding 153 is lifted.

2 General Constructions

We start with a straightforward observation.

Theorem 1. *Let $\mathcal{F}_1$ (resp. $\mathcal{F}_2$) be a $(t_1 \mod q)$-arc (resp. $(t_2 \mod q)$-arc) in $\mathrm{PG}(r,q)$. Then $\mathcal{F}_1+\mathcal{F}_2 \pmod q$ is a $(t \mod q)$ arc with $t=t_1+t_2 \pmod q$, provided all points in the sum have multiplicity at most t. In particular, the sum of t (not necessarily different) hyperplanes is a $(t \mod q)$-arc.*

The next construction is less obvious.

Theorem 2. *Let $\mathcal{F}_0$ be a $(t \mod q)$-arc in a hyperplane $H \cong \mathrm{PG}(r-1,q)$ of $\Sigma = PG(r,q)$. For a fixed point $P \in \Sigma \setminus H$, define an arc $\mathcal{F}$ in Σ as follows:*
– $\mathcal{F}(P)=t$;
– for each point $Q \neq P$: $\mathcal{F}(Q)=\mathcal{F}_0(R)$ where $R=\langle P,Q\rangle \cap H$.
Then the arc $\mathcal{F}$ is a $(t \mod q)$-arc in $\mathrm{PG}(k-1,q)$ of size $q|\mathcal{F}_0|+t$.

We call the $(t \mod q)$-arcs obtained by Theorem 2 *lifted arcs* and the point P – a *lifting point.*

It turns out that the lifting points of a $(t \mod q)$ arc form a subspace.

Lemma 1. *Let the arc $\mathcal{F}$ be lifted from the points P and Q, $P \neq Q$. Then $\mathcal{F}$ is also lifted from any point on the line PQ. In particular, the lifting points of a $(t \mod q)$-arc form a subspace S.*

The sum of t hyperplanes can be viewed as the sum of lifted arcs. Remarkably, we do not know an example of a $(t \mod q)$-arc in $\mathrm{PG}(r,q)$, with $r \geq 3$, that is not the sum of lifted arcs. It turns out that if in the geometry $\mathrm{PG}(r,q)$ there exist only lifted $(t \mod q)$-arcs then every $(t \mod q)$-arc in $\mathrm{PG}(r',q))$, $r' \geq r$, is also lifted.

Theorem 3. *Let $\mathcal{K}$ be a $(t \mod q)$-arc in $\mathrm{PG}(r,q)$ such that the restriction $\mathcal{K}|_H$ to every hyperplane H of $\mathrm{PG}(r,q)$ is also lifted. Then $\mathcal{K}$ is a lifted arc.*

In the plane case, non-trivial $(t \mod q)$-arcs can be constructed as σ-duals of certain blocking sets. Let $\mathcal{K}$ be a multiset in Σ. Consider a function σ such that $\sigma(\mathcal{K}(H))$ is a non-negative integer for all hyperplanes H. The multiset

$$\widetilde{\mathcal{K}}^{\sigma} : \begin{cases} \mathcal{H} & \rightarrow \quad \mathbb{N}_0 \\ H & \mapsto \quad \sigma(\mathcal{K}(H)) \end{cases} \tag{2}$$

in the dual space $\widetilde{\Sigma}$ is called the σ-*dual* of $\mathcal{K}$. If σ is a linear function, the parameters of $\widetilde{\mathcal{K}}^{\sigma}$, as well as its spectrum, are easily computed from the parameters and the spectrum of $\mathcal{K}$ (cf. [1,8]).

Theorem 4. *[6,7] Let $\mathcal{F}$ be a $(t \mod q)$-arc in $\mathrm{PG}(2,q)$ of size $mq+t$. Then the arc $\mathcal{F}^{\sigma}$ with $\sigma(x)=(x-t)/q$ is a $((m-t)q+m, m-t)$-blocking set in the dual plane. Moreover the multiplicities of the lines with respect to this blocking set belong to $\{m-t, m-t+1, \ldots, m\}$.*

3 $(2 \mod q)$-arcs

Let us note that an $(1 \mod q)$ arc is projective and hence either a hyperplane or the complete space [2,3]. For $t=2$ and odd $q \geq 5$, the $(t \mod q)$-arcs were characterized by Maruta [7]. These are the following:

(I) a lifted arc from a 2-line; such an arc has $2q+2$ points and there exist two possibilities

  (I-1) a double line, or

  (I-2) a sum of two different lines;

(II) a lifted arc from a $(q+2)$-line; such a line has i double points, $q-2i+2$ single points and $i-1$ 0-points, where $i=1,\ldots,\frac{q+1}{2}$; we say that such an arc is of type (II-i) if it is lifted from a line with i double points;

(III) a lifted arc from a $(2q+2)$-line, or, which is the same, the sum of two copies of the same plane;

(IV) an exceptional $(2 \mod q)$-arc for q odd; it consists of the points of an oval, a fixed tangent to this oval, and two copies of each internal point of the oval.

The following lemma is proved by investigating in some detal the types of lines in the different plane $(2 \mod q)$-arcs.

Lemma 2. *Let $\mathcal{K}$ be a $(2 \mod q)$-arc in $\mathrm{PG}(3,q)$, q odd, and let there exist a plane π such that $\mathcal{K}|_{\pi}$ is of type (IV). Then $\mathcal{K}$ is a lifted arc.*

Now we proceed by induction on the dimension. Again by Theorem 3 we get that every $(2 \mod q)$-arc in a geometry of dimension at least 3 is lifted.

Theorem 5. *Let $\mathcal{K}$ be a $(2 \mod q)$-arc in $\mathrm{PG}(r,q)$, q odd, $r \geq 3$. Then $\mathcal{K}$ is a lifted arc. In particular, every $(2 \mod q)$-arc in $\mathrm{PG}(r,q)$, $r \geq 2$, has a hyperplane in its support.*

Remark. Theorem 5 provides alternative proof of Maruta's theorem on the extendability of codes with weights $-2, -1, 0 \pmod{q}$ [10]. The existence of such a code is equivalent to that of an arc $\mathcal{K}$ which is 2-quasidivisible modulo q. It was pointed out in [6,7] that for every t-quasidivisible arc $\mathcal{K}$ in Σ it is possible to define uniquely a $(t \mod q)$- arc $\widetilde{\mathcal{K}}$ in the dual geometry. If $\widetilde{\mathcal{K}}$ contains a hyperplane in its support then $\mathcal{K}$ is extendable. This is the fact established in Theorem 5.

4 $(3 \mod q)$-arcs

For values of t larger than 2 complete classification seems out of reach. However, it is still possible to obtain partial results on the structure of such arcs. In this section we classify some small $(3 \mod 5)$-arcs in $\mathrm{PG}(2,5)$. Due to Theorem 4, the classification of such arcs is equivalent to the classification of certain blocking sets with an additional restriction on the line multiplicities.

Arcs of cardinality 18. These correspond to the sum of three not necessarily different lines in various mutual positions. It is an easy check that there exist for $(3 \mod 5)$-arcs of cardinality 18 [6].

Arcs of cardinality 23. These arcs correspond to $(9,1)$-blocking sets with lines of multiplicity 1, 2, 3, 4. The only possibility is the projective triangle. Dualizing we get a (3 mod 5-arc in which the 2-points form a complete quadrangle, the intersections of the diagonals are 3-points and the intersections of the diagonals with the sides of the quadrangle are 1-points.

Arcs of cardinality 28. The only $(3 \mod 5)$-arc of cardinality 28 has six 3-points forming an oval and ten 1-points that are the internal points to this oval.

Arcs of cardinality 33. If $\mathcal{F}$ is such an arc then $\mathcal{F}^\sigma$ is a $(21,3)$-blocking set with line multiplicities $3,4,5,6$. Again such a blocking set cannot have points of multiplicity 3 or larger since this would impose lines of multiplicity larger than 6 in $\mathcal{F}$. Denote by Λ_i the number of points of multiplicity i. Constructions are possible for $\Lambda_2 = 0,1,2$ constructions are possible. In such case, $\mathcal{F}^\sigma$ is one of the following:

(1) the complements of the seven non-isomorphic $(10,3)$-arcs; $\Lambda_2 = 0$;

(2) the complement of the $(11,3)$-arc with four external lines and a double point a point not on an external line, $\Lambda_2 = 1$;

(3) one double point which forms an oval with five of the 0-points; the tangent in the 2-point is a 3-line, $\Lambda_2 = 1$;

(4) $\mathrm{PG}(2,5)$ minus a triangle with vertices of multiplicity $2,2,1$; $\Lambda_2 = 2$.

Arcs of cardinality 38. The $(3 \mod 5)$-arcs of cardinality 38 can be derived from the (27,4)-blocking sets with line multiplicities $4,5,6,7$ in $\mathrm{PG}(2,5)$. Such a blocking set does not have 3-points and the number. Furthermore, if there exist three collinear 2-points then $\Lambda_2 = 3$ and the corresponding line is a 7-line.

There exist a lot of such blocking sets and, consequently, $(3 \mod 5)$-arcs of cardinality 38. In all cases, such arcs have a 13-line with a 0-point or an 8-line of type $(2,2,2,2,0,0)$, $(2,2,2,1,1,0)$ or $(3,3,2,0,0,0)$.

For instance, in the case of $\Lambda_2 = 0$ the blocking set consists of all points in the plane minus four points in general position. The corresponding $(3 \mod 5)$-arc has a line of type $(2,2,2,1,1,0)$. In the case $\Lambda_2 = 6$ the 2-points form an oval. Th external points to this oval have to be blocked at least four times by the fifteen 1-points. An easy counting gives that we should take necessarily the ten internal points plus five external points. But now the six tangents cannot be blocked twice by six points not on the oval. The remaining cases are treated using similar arguments.

Now we state our main result for this section.

Theorem 6. *Every* $(3 \mod 5)$*-arc* $\mathcal{F}$ *in* $\mathrm{PG}(3,5)$ *with* $|\mathcal{F}| \leq 158$ *is a lifted arc. In particular,* $|\mathcal{F}| = 93, 118$*, or* 143*.*

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