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Nonexistence of $(9, 112, 4)$ and $(10, 224, 5)$ binary orthogonal arrays [1]

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Abstract. Our approach is a natural continuation of the algorithm which is presented in [3]. We prove the nonexistence of BOAs of parameters (length, cardinality, strength) $= (9, 7.2^4 = 112, 4)$ and $(10, 7.2^5 = 224, 5)$, resolving two cases where the existence was undecided up to now.

1 Introduction

Let $H(n, 2)$ be the binary Hamming space of dimension n and the usual Hamming distance $d(x, y)$ between every two points $x, y \in H(n, 2)$. An orthogonal array (OA) of strength τ and index λ in $H(n, 2)$ (or binary orthogonal array, BOA), consists of the rows of an $M \times n$ matrix C with the property that every $M \times \tau$ submatrix of C contains all ordered τ-tuples of $H(\tau, 2)$, each one exactly $\lambda = M/2^\tau$ times as rows.

Let $C \subset H(n, 2)$ be an (n, M, τ) BOA. The distance distribution of C with respect to $c \in H(n, 2)$ if the $(n+1)$-tuple $w = w(c) = (w_0(c), w_1(c), \dots, w_n(c))$, where $w_i(c) = |\{x \in C | d(x, c) = i\}|$, $i = 0, \dots, n$. All feasible distance distributions of BOA of parameters (n, M, τ) can be computed effectively for relatively small n and τ as shown in [1] (see also [3]).

For fixed n, M and $\tau \le n$ we denote by $P(n, M, \tau)$ and $Q(n, M, \tau)$ the sets of all possible distance distributions of a (n, M, τ) BOA with respect to internal point and external point respectively. Denote also $W(n, M, \tau) = P(n, M, \tau) \cup Q(n, M, \tau)$. Here we start with the sets $P(n, M, \tau)$, $Q(n, M, \tau)$ and $W(n, M, \tau)$ which are obtained after applying the distance distributions algorithm from [3].

In this paper we present a natural generalization of the algorithm in [3] which again works on the sets $P(n, M, \tau)$, $Q(n, M, \tau)$ and $W(n, M, \tau)$ utilizing new connections between related BOAs. During the implementation of general algorithm these sets are changed by ruling out some distance distributions.

In Section 2 we prove several new relations between distance distributions of arrays under consideration and their relatives. As in [3] this imposes significant

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constraints on the targeted BOAs and allows us to collect rules for removing distance distributions from the sets $P(n,M,\tau)$, $Q(n,M,\tau)$ and $W(n,M,\tau)$. The logic of our algorithm and two new nonexistence results are described in Section 3.

2 Further relations between distance distributions of (n,M,τ) BOA and its derived BOAs

Let n, M and $3 \leq \tau < n+1$ be fixed. Let $C \subset H(n,2)$ be an (n,M,τ) BOA with sets of distance distributions $P(n,M,\tau)$, $Q(n,M,\tau)$ and $W(n,M,\tau)$ after [3]. For every $W \in W(n,M,\tau)$ we know all remaining couples (W,W') and for every such couple we have an uniquely determined corresponding couple (Y,X) which is obtained as a solution of system (2) from [3, Theorem 3]. So, for every $W \in W(n,M,\tau)$ we know all remaining triples (W',X,Y) which are not ruled out in [3]. Without loss of generality (as in [3]), $W \in W(n,M,\tau)$ is the distance distribution of C with respect to $c = \mathbf{0} \in H(n,2)$ and in (W',Y,X) we have: $W' = (w'_0, w'_1, \ldots, w'_{n-1}) \in W(n-1,M,\tau)$ – the distance distribution of C' with respect to $c' = \mathbf{0} \in H(n-1,2)$, $Y = (y_0, y_1, \ldots, y_{n-1}) \in W(n-1, M/2, \tau-1)$ – the distance distribution of C_0 with respect to c' and $X = (x_1, x_2, \ldots, x_n) \in W(n-1, M/2, \tau-1)$ – the distance distribution of C_1 with respect to c', where the BOA C_0 (C_1, respectively) is obtained from the rows of C with first coordinate 0 (1, respectively) after removing that first coordinate (see Fig.1 bellow).

We consider the BOAs C_0 and C_1 in the role of C and apply for both the DDA (part one), i.e. we removing the first column of C_0 and C_1 together and obtain BOAs A_0, A_1, B_0 and B_1 which distance distributions denote by R, Z, U and V, respectively. All BOAs and their distance distributions are illustrated of the Fig. 1 bellow. We apply Theorem 3 from [3] for C_0 and its derived BOAs (C'_0, A_0 and A_1) and for C_1 and its derived (C'_1, B_0 and B_1) together to obtain distance distributions $R, Z, U, V \in W(n-2, M/4, \tau-2)$.

Lemma 1. *The nonnegative integer numbers r_i, z_i, u_i, v_i, $i = 0, 1, \ldots, n-1$, satisfy the following system of linear equations*

$$\left| \begin{array}{l} z_i + r_i = y_i,\ i = 1, 2, \ldots, n-2 \\ z_{i+1} + r_i = y'_i,\ i = 0, 1, \ldots, n-2 \\ r_0 = y_0,\ z_{n-1} = y_{n-1} \\ v_i + u_i = x_{i+1},\ i = 1, 2, \ldots, n-2 \\ v_{i+1} + u_i = x'_{i+1},\ i = 0, 1, \ldots, n-2 \\ u_0 = x_1,\ v_{n-1} = x_n \end{array} \right. . \tag{1}$$

$|\ C'-(n-1,M,\tau),\ W'$

$|\ C''-(n-2,M,\tau),\ W''$

0	0	$R=(r_0,r_1,\dots,r_{n-2})$	
$\vdots$	$\vdots$	$A_0-(n-2,M/4,\tau-2)$	
0	0		$C_0,\ C_0'$
0	1	$Z=(z_1,z_2,\dots,z_{n-1})$	$Y,\ Y'$
$\vdots$	$\vdots$	$A_1-(n-2,M/4,\tau-2)$	
0	1		
1	0	$U=(u_0,u_1,\dots,u_{n-2})$	
$\vdots$	$\vdots$	$B_0-(n-2,M/4,\tau-2)$	
1	0		$C_1,\ C_1'$
1	1	$V=(v_1,v_2,\dots,v_{n-1})$	$X,\ X'$
$\vdots$	$\vdots$	$B_1-(n-2,M/4,\tau-2)$	
1	1		

Fig.1

$|\ C',\ W'$

$|\ C'',\ W''$

0	0	R	
$\vdots$	$\vdots$	A_0	
0	0		$D_0,\ D_0'$
1	0	U	$G,\ G'$
$\vdots$	$\vdots$	B_0	
1	0		
0	1	Z	
$\vdots$	$\vdots$	A_1	
0	1		$D_1,\ D_1'$
1	1	V	$H,\ H'$
$\vdots$	$\vdots$	B_1	
1	1		

Fig.2

We next apply the permutation $(0\to 1, 1\to 0)$ in the first column (see the comment before Theorem 10 in [3]) to obtain $C_0^{1,0}$ and $C_1^{1,0}$ with parameters $(n-1, M/2, \tau-1)$ and distance distributions $\widehat{Y}$ and $\widehat{X}$, respectively. The by Theorem 10 and Corollary 11 from [3] the BOAs $C_0^{1,0}$ and $C_1^{1,0}$ have distance distributions $\widehat{Y}=(z_1, z_2+r_0,\dots,z_{n-1}+r_{n-3},r_{n-2})$ and $\widehat{X}=(v_1,v_2+u_0,\dots,v_{n-1}+u_{n-3},u_{n-2})$, respectively.

As we discuss in the beginning of this section, for every $W\in W(n,M,\tau)$ we know the all remaining triples (W',Y,X) and for every such triple we have the sets $\{(Y,Y',R,Z)\}$ and $\{(X,X',U,V)\}$ of all possible distance distributions of the relatives BOAs which can obtain from the considering BOA C with this distance distribution $W\in W(n,M,\tau)$.

Now, for fixed $W\in W(n,M,\tau)$ we have a unique relation (W',Y,X)-(Y,Y',R,Z)-(X,X',U,V) (see Fig. 1). Notice that the obtained BOA C'' with parameters $(n-2,M,\tau)$ has distance distribution $W''=(w_0'',w_1'',\dots,w_{n-2}'')=(r_0+u_0+z_1+v_1, r_1+u_1+z_2+v_2,\dots,r_{n-2}+u_{n-2}+z_{n-1}+v_{n-1})\in W(n-2,M,\tau)$.

At the next step we reorder the rows of C' (simultaneously reordering the rows of the whole C) as we first take the rows with first coordinate 0, then we put the rows with first coordinate 1, respectively and remove that first coordinate. We again obtain C'' with the same distance distribution W'', but the derived BOAs with parameters $(n-1,M/2,\tau-1)$ are new. Let we denote them by D_0, D_1, D_0' and D_1' and let their distance distributions be G, H, G' and H', respectively. Furthermore, we again have the BOAs A_0, A_1, B_0 and B_1 with the same distance distributions $R,Z,U,V\in W(n-2,M/4,\tau-2)$ All BOAs (in this step) and their distance distributions are illustrated on the Fig. 2 above. We continue with description of the distance distributions D_0, D_1, D_0' and D_1' using the numbers r_i, z_i, u_i, v_i, $i=0,1,\dots,n-1$.

Theorem 2. *D_0 and D_1 are BOAs of parameters $(n-1, M/2, \tau-1)$ and distance distributions $G = (g_0, g_1, \ldots, g_{n-1}) = (r_0, r_1 + u_0, \ldots, r_{n-2} + u_{n-3}, u_{n-2})$ and $H = (h_1, h_2, \ldots, h_n) = (z_1, z_2 + v_1, \ldots, z_{n-1} + v_{n-2}, v_{n-1})$, i.e. $G, H \in W(n-1, M/2, \tau-1)$.*

Proof. The number of the points of D_0 at distance 0 from c' is r_0 (coming only from A_0), the number of the points of D_0 at distance i, $1 \le i \le n-2$, from c' is $r_i + u_{i-1}$ (as union of the points of A_0 at distance i from c' with the points of B_0 at distance $i-1$ from c'), and, finally, the number of the points of D_0 at distance $n-1$ from c' is u_{n-2} (coming only from B_0). Therefore the distance distribution of D_0 with respect to c' is $G = (g_0, g_1, \ldots, g_{n-1}) = (r_0, r_1 + u_0, \ldots, r_{n-2} + u_{n-3}, u_{n-2})$. Similarly the distance distribution of D_1 with respect to c' is $H = (h_1, h_2, \ldots, h_n) = (z_1, z_2 + v_1, \ldots, z_{n-1} + v_{n-2}, v_{n-1})$. □

Corollary 3. *a) The distance distribution G is ruled out if some of the related distance distributions $\overline{G}$, $\widehat{G}$ and $\overline{\widehat{G}}$ does not belong to $W(n-1, M/2, \tau-1)$;*

b) The distance distribution H is ruled out if some of the related distance distributions $\overline{H}$, $\widehat{H}$ and $\overline{\widehat{H}}$ does not belong to $W(n-1, M/2, \tau-1)$.

Theorem 4. *D_0' and D_1' are BOAs of parameters $(n-2, M/2, \tau-1)$ and distance distributions with respect to $c'' = \mathbf{0}'' \in H(n-2, 2)$ are $G' = (g_0', g_1', \ldots, g_{n-2}') = (r_0 + u_0, r_1 + u_1, \ldots, r_{n-2} + u_{n-2})$ and $H' = (h_1', h_2', \ldots, h_{n-1}') = (z_1 + v_1, z_2 + v_2, \ldots, z_{n-1} + v_{n-1})$, respectively, i.e. G', $H' \in W(n-2, M/2, \tau-1)$.*

Corollary 5. *a) The distance distribution G' is ruled out if some of the related distance distributions $\overline{G'}$, $\widehat{G'}$ and $\overline{\widehat{G'}}$ does not belong to $W(n-2, M/2, \tau-1)$;*

b) The distance distribution H' is ruled out if some of the related distance distributions $\overline{H'}$, $\widehat{H'}$ and $\overline{\widehat{H'}}$ does not belong to $W(n-2, M/2, \tau-1)$.

Further, we remove the second column of C to obtain a BOA C_2' with parameters $(n-1, M, \tau)$. Let $\tilde{W}'$ be the distance distribution of C_2' with respect to c'.

Theorem 6. *a) We have $\tilde{W}' = (r_0 + z_1, u_0 + r_1 + v_1 + z_2, \ldots, u_{n-3} + r_{n-2} + v_{n-2} + z_{n-1}, u_{n-2} + v_{n-1}) \in W(n-1, M, \tau)$.*

b) The distance distribution of $(C_2')^{1,0}$ with respect to c' is $\widehat{\tilde{W}'} = (u_0 + v_1, r_0 + u_1 + z_1 + v_2, \ldots, r_{n-3} + u_{n-2} + z_{n-2} + v_{n-1}, r_{n-2} + z_{n-1}) \in W(n-1, M, \tau)$.

Corollary 7. *The distance distribution $\tilde{W}'$ is ruled out if some of the related distance distributions $\overline{\tilde{W}'}$, $\widehat{\tilde{W}'}$ and $\overline{\widehat{\tilde{W}'}}$ does not belong to $W(n-1, M, \tau)$.*

Next, we consider the effect of the permutation $(0 \to 1, 1 \to 0)$ in the first two columns (simultaneously). Denote the new BOA with $\tilde{C}$.

Theorem 8. *The distance distribution of* $\tilde{C}$ *with respect to* c *is* $\tilde{W} = (v_1, u_0 + z_1 + v_2, r_0 + u_1 + z_2 + v_3, \ldots, r_{n-4} + u_{n-3} + z_{n-2} + v_{n-1}, r_{n-3} + u_{n-2} + z_{n-1}, r_{n-2}) \in W(n, M, \tau)$.

Corollary 9. *The distance distribution* $\tilde{W}$ *is ruled out if some of the related distance distributions* $\overline{\tilde{W}}$, $\widehat{\tilde{W}}$ *and* $\widehat{\overline{\tilde{W}}}$ *does not belong to* $W(n, M, \tau)$.

After all above checks, for every survival $W \in W(n, M, \tau)$ we have attached triples (W', Y, X)-(Y, Y', R, Z)-(X, X', U, V). We now free the cut of the second column and thus consider all possible $n - 1$ cuts of columns of C'. These cuts produce all possible pairs $\{(Y, Y', R, Z)\}$ - $\{(X, X', U, V)\}$. Let

$$(z_0^{(i)} = 0, z_1^{(i)}, \ldots, z_{n-2}^{(i)}, z_{n-1}^{(i)}; r_0^{(i)}, r_1^{(i)}, \ldots, r_{n-2}^{(i)}, r_{n-1}^{(i)} = 0), \quad i = 1, \ldots, s,$$

$$(v_0^{(j)} = 0, v_1^{(j)}, \ldots, v_{n-2}^{(j)}, v_{n-1}^{(j)}; u_0^{(j)}, u_1^{(j)}, \ldots, u_{n-2}^{(j)}, v_{n-1}^{(j)} = 0), \quad j = 1, \ldots, t,$$

are all solutions of system (1) for the triple (W', Y, X) that remain after applying Corollary 3, Corollary 5, Corollary 7 and Corollary 9.

Theorem 10. *The nonnegative integers* $k_{i,j}$, $i = 1, \ldots, s; j = 1, \ldots, t$, *satisfy the following system of linear equations*

$$\left| \begin{array}{l} \sum\limits_{i,j} k_{i,j} = n, \ \sum\limits_{i,j} k_{i,j} z_1^{(i,j)} = y_1, \ \sum\limits_{i,j} k_{i,j} z_2^{(i,j)} = 2y_2, \ \ldots, \sum\limits_{i,j} k_{i,j} z_{n-1}^{(i,j)} = n y_{n-1} \\ \sum\limits_{i,j} k_{i,j} v_1^{(i,j)} = x_2, \ \sum\limits_{i,j} k_{i,j} v_2^{(i,j)} = 2x_3, \ \ldots, \sum\limits_{i,j} k_{i,j} v_{n-1}^{(i,j)} = n x_n \end{array} \right. . \tag{2}$$

Proof. This follows from counting in two ways the number of the ones in the i-blocks of C_0 and C_1, respectively (see Theorem 13 in [3]). □

Corollary 11. *The triple* (W', Y, X) *is ruled out if the system (2) does not have solutions.*

3 Two new nonexistence results

Let $C = (n, M, \tau)$ be a BOA of targeted parameters, where $\tau \geq 3$. First we apply the algorithm from [3], and obtain reduced sets P, Q and W for C and its relatives. We also have, for every $W \in W(n, M, \tau)$, the sets of all feasible triples (W', Y, X). For every such triple we find the corresponding sets $\{(Y, Y', R, Z)\}$ and $\{(X, X', U, V)\}$ as explained in the previous section. We continue with the implementation of new observations and organize them to work together as follows.

For every fixed W-(W', Y, X)-(Y, Y', R, Z)-(X, X', U, V) we apply Theorems 2, 4, 6, 8 and 10 in every row separately from left to right to reduce the sets P,

Q and W. Of course, this process is fueled with information from the columns (starting from the bottom end) according to Corollaries 3, 5, 7, 9 and 11. The algorithm stops when no new rulings out are possible. An entry at the right end, showing that some of the sets $P(n, M, \tau)$, $Q(n, M, \tau)$ and $W(n, M, \tau)$ is empty, means nonexistence of the corresponding BOA.

For a putative BOAs with parameters $(9, 112, 4)$ and $(10, 224, 5)$ we end with empty $W(9, 112, 4)$ and $W(10, 224, 5)$.

Theorem 12. *There exist no binary orthogonal arrays of parameters* $(9, 112, 4)$ *and* $(10, 224, 5)$.

The second results follows also from the first one and the coexistence of $(n, N, 2k)$ and $(n+1, 2N, 2k+1)$ (see [5], [4, Theorem 2.24]).

The nonexistence results of Theorem 12 give exact values for the function $L(n, \tau)$ – the minimum possible index λ of an $(n, M = \lambda 2^\tau, \tau)$ binary orthogonal array. We have $L(n, \tau) = 8$ instead of $7 \leq L(n, \tau) \leq 8$ for $(n, \tau) = (9, 4)$ and $(10, 5)$.

All calculations in this paper were performed by programs in Maple. All results can be seen at [6]. All programs are available upon request [6].

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