*Fifteenth International Workshop on Algebraic and Combinatorial Coding Theory*
*June 18-24, 2016, Albena, Bulgaria* *pp. 206–208*

# Linear Codes Close to the Griesmer Bound and the Related Geometric Structures [1]

Ivan Landjev ivan@math.bas.bg
Institute of Mathematics and Informatics, BAS
8 Acad. G. Bonchev str., 1113 Sofia, Bulgaria
Assia Rousseva assia@fmi.uni-sofia.bg
Faculty of Mathematics and Informatics, Sofia University, 5 James Bourchier Blvd., 1126 Sofia, Bulgaria

**Abstract.** We investigate the following version of the main problem of coding theory: Given the integer $k$ and the prime power $q$, what is the value of

$$t_q(k) := \min_d n_q(k,d) - g_q(k,d).$$

The Griesmer bound for a linear $[n, k, d]_q$-code is a lower bound on the length $n$ as a function of $q$, $k$, and $d$ [3, 5]:

$$n \geq g_q(k,d) := \sum_{i=0}^{k-1} \lceil \frac{d}{q^i} \rceil.$$

It is known that for fixed $q$ and $k$ Griesmer codes do exist for sufficiently large $d$ [2, 4]. In fact, it follows by the Belov-Logachev-Sandimirov Theorem [1] that this is true for all $d \geq (k-2)q^{k-1} + 1$ On the other hand, a less known result by Dodunekov [2] says that for fixed $q$ and $d$ and $k \to \infty$

$$n_q(k,d) - g_q(k,d) \to \infty.$$

The following question can be viewed as a version of the main problem of coding theory:

Given the integer $k$ and the prime power $q$, what is the exact value of

$$t_q(k) := \min_d n_q(k,d) - g_q(k,d),$$

or, in other words,what is the smallest value of $t$, such that there exists a $[t + g_q(k,d), k, d]_q$-code.

It is well-known that $t_q(2) = 0$ [4]. The problem is open even for $k = 3$ and was asked by S. Ball in the following way:

[1]This research was supported by the Scientific Research Fund of Sofia University under Contract 29/11.04.2016.

For a fixed $n-d$, is there always a 3-dimensional code meeting the Griesmer bound (maybe a constant or $\log q$ away)?

Numerical evidence shows that $t_q(3) = 1$ for $q \leq 19$ and $t_q(3) \leq 2$ for all $q \leq 25$. For larger dimensions we know that: $t_3(4) = 1, t_4(4) = 1, t_5(4) = 2$.

Let

$$d = sq^{k-1} - \lambda_{k-2}q^{k-2} - \ldots - \lambda_1 q - \lambda_0, \tag{1}$$

where $0 \leq \lambda_i < q$. Now it is easily checked that

$$g_q(k,d) = sv_k - \lambda_{k-2}v_{k-1} - \ldots - \lambda_1 v_2 - \lambda_0 v_1, \tag{2}$$

where $v_i = (v^i - 1)/(v-1)$. Now a Griesmer $[n,k,d]_q$-code can be associated with an $(n,w)$-arc in $\mathrm{PG}(k-1,q)$ with $n = g_q(k,d)$ and

$$w = sv_{k-1} - \lambda_{k-2}v_{k-2} - \ldots - \lambda_1 v_1.$$

The complement of such an arc is a minihyper with parameters

$$(\lambda_{k-2}v_{k-1} + \ldots + \lambda_1 v_2 + \lambda_0 v_1, \lambda_{k-2}v_{k-2} - \ldots - \lambda_1 v_1). \tag{3}$$

Now our problem can be formulated as follows:

Given $d$ by (1), find the smallest value of $t$ for which there exists a $(g_q(k,d)+t, g_q(k,d)+t-d)$-arc in $\mathrm{PG}(k-1,q)$ for all $d$.

In terns of minihypers it can also be formulated in the following way:

Given $d$ by (1), find the smallest value of $t$ for which there exists a minihyper with parameters

$$(\lambda_{k-2}v_{k-1} + \ldots + \lambda_1 v_2 + \lambda_0 v_1 - t, \lambda_{k-2}v_{k-2} - \ldots - \lambda_1 v_1 - t)$$

in $\mathrm{PG}(k-1,q)$.

Now we have the following theorem.

**Theorem.** Let $d$ be given by (1) and let the multiset $[\mathcal{F}$ be the sum of $\lambda_{k-2}$ hyperplanes, $\lambda_{k-3}$ hyperlines etc. $\lambda_1$ lines, $\lambda_0$ points. Define the multiset $\mathcal{F}'$ by

$$\mathcal{F}'(x) = \begin{cases} \mathcal{F}(x) & \text{if } \mathcal{F}(x) \leq s, \\ s & \text{if } \mathcal{F}(x) > s. \end{cases}$$

Let $N = |\mathcal{F}|$ and $N' = |\mathcal{F}'|$. If $\mathcal{F} - \mathcal{F}'$ is an $(N - N', u)$-arc then $t \leq u$.

For 3-dimensional codes over square fields we have the following theorem which gives probably a very weak bound on $t_q(3)$.

**Theorem.** $t_q(3) \leq \sqrt{q}$.

We also present some recursive bounds on $t_q(k)$ for small values of $k$.

## References

[1] B. I. Belov, V. N. Logachev, V. P. Sandimirov, Construction of a class of linear binary codes achieving the Varshamov-Griesmer bound, *Probl. Inf. Transm.* **10**(3)(1974), 211–217.

[2] S. Dodunekov, Optimal Codes, DSc Thesis, Sofia, 1985.

[3] J. H. Griesmer, A bound for error-correcting codes, *IBM J. Res. Develop.* **4**(1960), 532–542.

[4] R. Hill, Optimal Linear Codes, Cryptography and Coding II (C. Mitchell ed.), Oxford Univ. Press (1992), 75–104.

[5] G. Solomon, J. J. Stiffler, Algebraically punctured cyclic codes, *inf. and Control* **8**(1965), 170–179.