Fifteenth International Workshop on Algebraic and Combinatorial Coding Theory
June 18-24, 2016, Albena, Bulgaria *pp. 193–199*

On the Probability of Error for Triangular Quadrature Amplitude Modulation [1]

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Abstract. We compute the exact value of error probability per symbol (SER) for triangular quadrature amplitude modulation (TQAM) scheme in the case of AWGN channel. Also, we derive an upper and a lower bounds for SER in this case. The results , as well as the simulations, show that the difference between the exact value of SER and the upper bound bound is constant. Hence the simple upper bound can be used in practice for evaluating the SER.

1 Introduction

Nowadays, in modern digital communication systems, high-order modulation is preferred for high-speed data transmission. One of the most popular modulation in commercial communication systems is square quadrature amplitude modulation (SQAM). SQAM scheme with its simple detection procedure is easy for implementation and demonstrates a good performance.

Recently, the triangular quadrature amplitude modulation (TQAM) was proposed. In TQAM constellation the signal points are vertexes of a lattice of equilateral triangles and the constellation is symmetric with respect to the origin. The comparison of TQAM with SQAM given in [3] shows that the former is more power efficient while preserves the low detection complexity of the latter. In [4] a general formula for calculating the average energy per symbol of the TQAM is derived and symbol error rate (SER) and bit error rate (BER) of the TQAM in the presence of additive white Gaussian noise (AWGN) is analyzed.

In the next section we give a brief description of TQAM. In Section 3 we derived the exact value of SER for 2^{2m}-TQAM constellations, uncoded case. In the last section we compare the obtained results with the lower and upper bounds given in [2].

[1]The research of the second author is supported by the National Science Fund of Bulgaria under Grant DFNI-I02/8.

2 TQAM constellation

In this paper we consider TQAM constellation of $M = 2^{2m}$ signal points placed in $L = 2^m$ rows parallel to real axis with L signal point in each row. The points form a lattice of equilateral triangles and the constellation is symmetric with respect to the origin. An example of 64-ary TQAM is given in Fig. 1.

The power gain of M-ary TQAM over M-ary SQAM in decibels [4] is

$$10 \log_{10} \left(\frac{8M-8}{7M-4} \right) \xrightarrow[M \to \infty]{} 0.5799 \text{ dB}$$

For $M = 16,\ 64,\ 256$ the power gain is 0.458, 0.5505 and 0.5726, respectively.

3 The SER in uncoded case

The L$^2-$TQAM constellation can be separated into seven types of detection regions $D_1, D_2, \ldots, D_7$. In this section we will calculate the probability of correct detection q_i for each of the regions D_i, $i = 1, \ldots, 7$. The number of detection regions of each type for L^2-TQAM is given in the next table.

D_1	D_2	D_3	D_4	D_5	D_6	D_7
$(L-2)^2$	$2(L-2)$	$L-2$	2	2	2	$L-4$

Note that in the case of 16-TQAM ($L = 4$) the region D_7 does not exist.

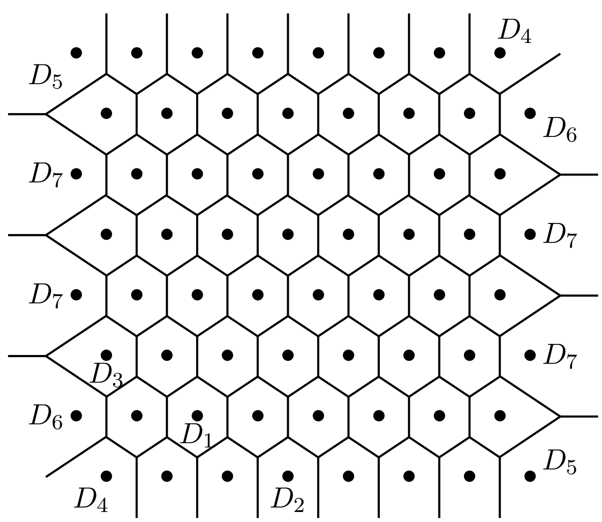

Figure 1. 64-TQAM constellation.

The lines that are boundaries of any region D_i have the following equations according to the coordinate system with the signal point in the region D_i as origin:

- $y = \dfrac{2d+x}{\sqrt{3}}$ the line left-above; $y = \dfrac{2d-x}{\sqrt{3}}$ the line right-above;
- $y = \dfrac{-x-2d}{\sqrt{3}}$ the line left-down; $y = \dfrac{x-2d}{\sqrt{3}}$ the line right-down;
- $x = \text{const}$ vertical line; $y = \text{const}$ horizontal line.

3.1 Region D_1 (hexagonal)

It consists of 4 congruent subregions. The right-above one is defined by

$$D_1 : \left\{ 0 \le x \le d; 0 \le y \le \frac{2d-x}{\sqrt{3}} \right\}$$

Recalling that $\Pr\left\{0 \le y \le \frac{2d-x}{\sqrt{3}}\right\} = \frac{1}{2}\mathrm{erf}\left(\frac{2d-x}{\sqrt{3}}\right)$ we conclude that the one forth of the probability of correct detection for D_1 is

$$\frac{1}{4}q_1 = \int_0^d \frac{1}{\sqrt{\pi N_0}} e^{-x^2/N_0} \mathrm{erf}\left(\frac{2d-x}{\sqrt{3n_0}}\right) dx,$$

Hence

$$q_1 = \frac{2}{\sqrt{\pi N_0}} \int_0^d e^{-x^2/N_0} \mathrm{erf}\left(\frac{2d-x}{\sqrt{3N_0}}\right) dx \tag{1}$$

3.2 Region D_2

These regions are symmetrically placed at the top and bottom of the constellation. Each of them is one side unbounded in y. Let us consider the region with negative values of y. It consists of two congruent parts and the right part is defined by $\left\{0 \le x \le d; \quad -\infty \le y \le \frac{2d-x}{\sqrt{3}}\right\}$. Obviously, it can be separated into two parts: rectangular $\{0 \le x \le d; \quad -\infty \le y \le 0\}$ and one forth of D_1. Thus $q_2/2 = q_1/4 + (1/2)\mathrm{erf}(d/\sqrt{N_0}).(1/2)$ or

$$q_2 = \frac{1}{2}q_1 + \frac{1}{2}\mathrm{erf}\left(\frac{d}{\sqrt{N_0}}\right) \tag{2}$$

3.3 Region D_3 (pentagonal)

The probability of correct detection is

$$q_3 = \frac{1}{2}q_1 + 2\,\mathrm{Pr}\left\{-2d \le x \le 0; \quad 0 \le y \le \frac{2d+x}{\sqrt{3}}\right\}$$

Changing $x \to -x$ we get

$$q_3 = \frac{1}{2}q_1 + \frac{1}{\sqrt{\pi N_0}} \int_0^{2d} e^{-\frac{x^2}{N_0}} \mathrm{erf}\left(\frac{2d-x}{\sqrt{3N_0}}\right) dx \tag{3}$$

3.4 Region D_4

This is infinite region that is union of D_2 and infinite triangle bounded by the lines $x = -d$ and $y\sqrt{3} = 2d + x$. Hence

$$\begin{aligned} q_4 &= q_2 + \mathrm{Pr}\left\{-\infty \le x \le -d; \quad -\infty \le y \le \frac{2d+x}{\sqrt{3}}\right\} \\ &= q_2 + \frac{1}{\sqrt{\pi N_0}} \int_{-\infty}^{-d} e^{-\frac{x^2}{N_0}} \frac{1}{2}\left(1 + \mathrm{erf}\left(\frac{2d+x}{\sqrt{3N_0}}\right)\right) dx \end{aligned}$$

Hence changing $x \to -x$ and using the definition of error function we have

$$q_4 = q_2 + \frac{1}{4}\left(1 - \mathrm{erf}\left(\frac{d}{\sqrt{N_0}}\right)\right) + \frac{1}{2\sqrt{\pi N_0}} \int_d^{\infty} e^{-\frac{x^2}{N_0}} \mathrm{erf}\left(\frac{2d-x}{\sqrt{3N_0}}\right) dx \tag{4}$$

3.5 Region D_5

D_5 is infinite rectangular $\{-\infty \le x \le d,\ -d\sqrt{3} \le y \le \infty\}$ with cut angle, the triangle $\Delta:\ \{-d \le x \le d,\ -d\sqrt{3} \le y \le (x-2d)/\sqrt{3}\}$. Therefore

$$q_5 = \frac{1}{2}\left(1 + \mathrm{erf}\left(\frac{d}{\sqrt{N_0}}\right)\right) \cdot \frac{1}{2}\left(1 + \mathrm{erf}\left(\frac{d\sqrt{3}}{\sqrt{N_0}}\right)\right) - q_\Delta, \quad \text{where}$$

$$\begin{aligned} q_\Delta &= \frac{1}{\sqrt{\pi N_0}} \int_{-d}^{d} e^{-\frac{x^2}{N_0}} \frac{1}{2}\left(\mathrm{erf}\left(\frac{d\sqrt{3}}{\sqrt{N_0}}\right) - \mathrm{erf}\left(-\frac{x-2d}{\sqrt{3N_0}}\right)\right) dx \\ &= \frac{1}{2}\mathrm{erf}\left(\frac{d\sqrt{3}}{\sqrt{N_0}}\right) \mathrm{erf}\left(\frac{d}{\sqrt{N_0}}\right) - \frac{1}{2\sqrt{\pi N_0}} \int_{-d}^{d} e^{-\frac{x^2}{N_0}} \mathrm{erf}\left(\frac{2d-x}{\sqrt{3N_0}}\right) dx \end{aligned}$$

Therefore

$$\begin{aligned} q_5 &= \frac{1}{4}\left(1+\operatorname{erf}\left(\frac{d}{\sqrt{N_0}}\right)+\operatorname{erf}\left(\frac{d\sqrt{3}}{\sqrt{N_0}}\right)\right)-\frac{1}{4}\operatorname{erf}\left(\frac{d\sqrt{3}}{\sqrt{N_0}}\right)\operatorname{erf}\left(\frac{d}{\sqrt{N_0}}\right) \\ &+ \frac{1}{2\sqrt{\pi N_0}}\int_{-d}^{d} e^{-\frac{x^2}{N_0}}\operatorname{erf}\left(\frac{2d-x}{\sqrt{3N_0}}\right)dx \end{aligned} \tag{5}$$

3.6 Region D_6

D_6 can be obtained from D_5 by substraction of the infinite triangle $\delta:\ \{-\infty \le x \le d,\ -\infty \le y \le \frac{x-2d}{\sqrt{3}}\}$, i.e., $q_6 = q_5 - q_\delta$, where

$$\begin{aligned} q_\delta &= \frac{1}{2\sqrt{\pi N_0}}\int_{-\infty}^{d} e^{-\frac{x^2}{N_0}}(1-\operatorname{erf}\left(-\frac{x-2d}{\sqrt{3N_0}}\right))\,dx \\ &= \frac{1}{4}\left(1+\operatorname{erf}\left(\frac{d}{\sqrt{N_0}}\right)\right)-\frac{1}{2\sqrt{\pi N_0}}\int_{-\infty}^{d} e^{-\frac{x^2}{N_0}}\operatorname{erf}\left(\frac{2d-x}{\sqrt{3N_0}}\right)dx \end{aligned}$$

Therefore

$$q_6 = q_5 - \frac{1}{4}\left(1+\operatorname{erf}\left(\frac{d}{\sqrt{N_0}}\right)\right)+\frac{1}{2\sqrt{\pi N_0}}\int_{-\infty}^{d} e^{-\frac{x^2}{N_0}}\operatorname{erf}\left(\frac{2d-x}{\sqrt{3N_0}}\right)dx \tag{6}$$

3.7 Region D_7

This region exists only for $L \ge 6$. It is an infinite rectangular with cut two angles: the symmetrically placed triangles τ and τ', i.e.,

$$D_7:\ \{-\infty \le x \le d,\ -d\sqrt{3} \le y \le d\sqrt{3}\} \setminus \{\tau \cup \tau'\}, \qquad \text{where}$$

$$\tau:\ \{-d \le x \le d,\ \frac{x-2d}{\sqrt{3}} \le y \le d\sqrt{3}\}; \quad \tau':\ \{-d \le x \le d,\ -d\sqrt{3} \le y \le \frac{x-2d}{\sqrt{3}}\}$$

Since τ and τ' have the same contribution to the probability and τ' coincides with Δ from Subsection E we can write

$$q_7 = \frac{1}{2}\left(1+\operatorname{erf}\left(\frac{d}{\sqrt{N_0}}\right)\right)\operatorname{erf}\left(\frac{d\sqrt{3}}{\sqrt{N_0}}\right)-2q_\Delta, \qquad \text{thus}$$

$$q_7 = \frac{1}{2}\left(1-\operatorname{erf}\left(\frac{d}{\sqrt{N_0}}\right)\right)\operatorname{erf}\left(\frac{d\sqrt{3}}{\sqrt{N_0}}\right)+\frac{1}{\sqrt{\pi N_0}}\int_{-d}^{d} e^{-\frac{x^2}{N_0}}\operatorname{erf}\left(\frac{2d-x}{\sqrt{3N_0}}\right)dx \tag{7}$$

Hence the probability q for a correct detection of the received signal point is

$$q = \frac{1}{L^2}[(L-2)^2 q_1 + 2(L-2)q_2 + (L-2)q_3 + 2(q_4+q_5+q_6) + (L-4)q_7]$$

and thus

$$SER = 1 - q. \tag{8}$$

The values q_i and SER in the case $L^2 = 16$ are given in Table 1.

Table 1: SER for regions of 16-TQAM

	D_1	D_2	D_3	D_4	D_5	D_6	SER	mult.
10dB	2.9535	2.1569	2.6473	1.6646	1.1739	1.6662	2.1716	$\times 10^{-1}$
11dB	2.1576	1.5508	1.9099	1.1912	0.8320	1.1916	1.5676	$\times 10^{-1}$
12dB	1.4544	1.030	1.2723	0.7876	0.5452	0.7876	1.0452	$\times 10^{-1}$
13dB	8.8660	6.1946	7.6721	4.7169	3.2394	4.7170	6.308	$\times 10^{-2}$
14dB	4.7653	3.2899	4.0841	2.4957	1.7015	2.4957	3.3609	$\times 10^{-2}$
15dB	2.1883	1.4955	1.8602	1.1308	7.6611	1.1308	1.5319	$\times 10^{-2}$
16dB	8.2558	5.5959	6.9720	4.2197	2.8441	4.2202	5.7449	$\times 10^{-3}$
17dB	2.4387	1.6423	2.0486	1.2360	0.8292	1.2356	1.6890	$\times 10^{-3}$
18dB	5.3007	3.5541	4.4376	2.6706	1.7871	2.6679	3.6591	$\times 10^{-4}$
19dB	7.8686	5.2607	6.5719	3.9492	2.6358	3.9458	5.4202	$\times 10^{-5}$
20dB	7.2504	4.8394	6.0508	3.3603	2.4213	3.6999	4.9640	$\times 10^{-6}$

4 Conclusion

In [2] we derived the following lower and upper bounds for the uncoded case:

$$e^{-\frac{16}{7L^2-4}\mathrm{SNR}_s} < \mathrm{SER} < e^{-\frac{12}{7L^2-4}\mathrm{SNR}_s}. \tag{9}$$

The upper bound has a behavior very similar to one of the exact value of SER versus signal/noise ratio in dB. This was the start point for us to find the following approximations.

$$\text{16TQAM:} \qquad \mathrm{SER} \approx e^{-\frac{12.75}{108}SNR\left(\frac{|SNR_{dB}-16|}{100}+0.645\right)} \tag{10}$$

$$\text{64TQAM:} \qquad \mathrm{SER} \approx e^{-\frac{12.70}{444}SNR\left(\frac{|SNR_{dB}-20|}{100}+0.79\right)} \tag{11}$$

$$\text{256TQAM:} \qquad \mathrm{SER} \approx e^{-\frac{12.65}{1788}SNR\left(\frac{|SNR_{dB}-28|}{100}+0.83\right)} \tag{12}$$

where SNR_{dB} is the signal/noise ratio in decibels and $SNR = 10^{\frac{SNR_{dB}}{10}}$.

Figure 2 demonstrates how good is the approximation for 256-TQAM.

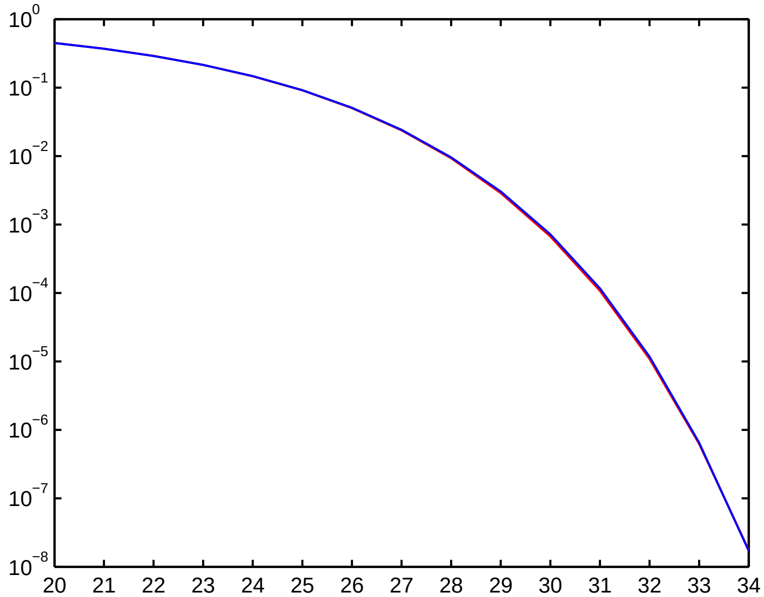

Figure 2. SER and approximation for 256-TQAM.

References

[1] S. Haykin: *Digital Communications*, John Wiley and Sons, N.Y. (1988).

[2] H. Kostadinov, L. Kraleva and N. L. Manev: New Bounds for Probability of Error of Coded and Uncoded TQAM in AWGN Channel, *BGSIAM* (2015).

[3] Sung-Joon Park: Triangular quadrature amplitude modulation, *IEEE Commun. Lett.*, vol. 11 (2007), no. 4, 292–294.

[4] Sung-Joon Park: Performance Analysis of Triangular Quadrature Amplitude Modulation in AWGN Channel. *IEEE Commun. Letters*, Vol. 16 (2012), 765–768.