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Universal Lower Bounds on Energy and LP-Extremal Polynomials for $(4, 24)$-Codes [1]

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Abstract. In this paper we introduce the framework for improvement of the universal lower bounds (ULB) on potential energy using the Delsarte-Yudin linear programming approach for polynomials. As a model example we consider the case of 24 points on $\mathbb{S}^3$.

1 Introduction

Let $\mathbb{S}^{n-1}$ denote the unit sphere in $\mathbb{R}^n$. A finite set $C \subset \mathbb{S}^{n-1}$ is called a *spherical code*. Given an (extended real-valued) function $h(t) : [-1, 1] \to [0, +\infty]$, the *h-energy* of a spherical code C is given by

$$E(C; h) := \sum_{x,y \in C, x \neq y} h(\langle x, y \rangle), \tag{1}$$

where $\langle x, y \rangle$ denotes the inner product of x and y. We are interested in lower bounds on energy of codes C with fixed cardinality $|C| = N$, referred to here as (n, N)-*codes*, $\mathcal{E}(n, N; h) := \inf\{E(C; h) : |C| = N, C \subset S^{n-1}\}$.

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Delsarte-Yudin's approach for finding such lower bounds is described as follows. Suppose the class $\mathcal{A}_{n,h}$ consists of all functions $f : [-1,1] \to \mathbb{R}$ s. t.

$$\mathcal{A}_{n,h} := \{f(t) \ : \ f(t) = \sum_{k=0}^{\infty} f_k P_k^{(n)}(t) \leq h(t), \quad f_k \geq 0, \quad k = 1, 2, \dots\}, \qquad (2)$$

where $\{P_k^{(n)}(t)\}$ are the Gegenbauer polynomials orthogonal on $[-1,1]$ with respect to a measure $(1-t^2)^{(n-3)/2}\,dt$ and normalized so that $P_k^{(n)}(1) = 1$. Then

$$\mathcal{E}(n, N; h) \geq \max_{f \in \mathcal{A}_{n,h}} \left(f_0 N^2 - f(1)N \right). \qquad (3)$$

Instead of solving the infinite linear program in the right-hand side of (3) one may restrict to a subspace $\Lambda \subset$C([-1,1]) (usually finite-dimensional), namely determining the quantity

$$\mathcal{W}(n, N, \Lambda; h) := \sup_{f \in \Lambda \cap A_{n,h}} N^2 (f_0 - f(1)/N). \qquad (4)$$

In [1] we derived *Universal Lower Bounds* (ULB) on energy by explicitly solving (4) when $\Lambda = \mathcal{P}_m$, the polynomials of degree at most $m \leq \tau(N, n)$ for certain $\tau(N, n)$. The goal of this article is to introduce a framework for solving the linear program in some cases when $m > \tau(N, n)$ and obtain improved ULB.

2 $1/N$-Quadrature rules and lower bounds for energy on subspaces

Thereafter we consider only absolutely monotone potentials h, that is functions $h(t)$, such that $h^{(k)}(t) \geq 0$, for every $t \in [-1,1]$ and every integer $k \geq 0$. An important ingredient in [1] is the notion of a $1/N$-quadrature over subspaces, which we briefly review. A finite sequence of ordered pairs $\{(\alpha_i, \rho_i)\}_{i=1}^k$, $-1 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_k < 1$, $\rho_i > 0$ for $i = 1, 2, \dots, k$, is said to define a $1/N$-*quadrature rule* over the subspace $\Lambda \subset C([-1,1])$ if

$$f_0 := \gamma_n \int_{-1}^{1} f(t)(1-t^2)^{(n-3)/2} dt = \frac{f(1)}{N} + \sum_{i=1}^{k} \rho_i f(\alpha_i), \ \gamma_n := \frac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{n-1}{2}\right)} \qquad (5)$$

is *exact* for all $f \in \Lambda$. The following theorem is found in [1].

Theorem 2.1 ([1], Theorems 2.3 and 2.6)**.** *Let $\{(\alpha_i, \rho_i)\}_{i=1}^k$ be a $1/N$-quadrature rule that is exact for a subspace $\Lambda \subset C([-1,1])$. If $f \in \Lambda \cap A_{n,h}$, then $\mathcal{E}(n, N; h) \geq N^2 \sum_{i=1}^k \rho_i f(\alpha_i)$ and*

$$\mathcal{W}(n, N, \Lambda; h) \leq N^2 \sum_{i=1}^{k} \rho_i h(\alpha_i). \qquad (6)$$

If there is some $f \in \Lambda \cap A_{n,h}$ such that $f(\alpha_i) = h(\alpha_i)$ for $i = 1, \ldots, k$, then equality holds in (6)*, which yields the universal lower bound*

$$\mathcal{E}(n, N; h) \geq N^2 \sum_{i=1}^{k} \rho_i h(\alpha_i). \tag{7}$$

Furthermore, in this case if $\Lambda' = \Lambda \bigoplus span \, \{P_j^{(n)} : j \in \mathcal{I}\}$ for some index set $\mathcal{I} \subset \mathbb{N}$ and the test functions (see [1, Theorems 2.6, 4.1])

$$Q_j^{(n)} := \frac{1}{N} + \sum_{i=1}^{k} \rho_i P_j^{(n)}(\alpha_i) \tag{8}$$

satisfy $Q_j^{(n)} \geq 0$ for $j \in \mathcal{I}$, then

$$\mathcal{W}(n, N, \Lambda'; h) = \mathcal{W}(n, N, \Lambda; h) = N^2 \sum_{i=1}^{k} \rho_i h(\alpha_i). \tag{9}$$

3 Levenshtein's framework and ULB

Of particular importance is the case when the subspace in Section 2 is $\mathcal{P}_m$. For this purpose we briefly introduce Levenshtein's framework (see [5]). First, we next recall two classical notions. The *Delsarte-Goethals-Seidel* lower bound $D(n, \tau)$ on the cardinality of spherical designs of strength τ is given by (cf. [4])

$$D(n, \tau) := \begin{cases} 2\binom{n+k-2}{n-1}, & \text{if } \tau = 2k-1, \\ \binom{n+k-1}{n-1} + \binom{n+k-2}{n-1}, & \text{if } \tau = 2k. \end{cases} \tag{10}$$

A close cousin, *Levenshtein's upper bound* $L(n, s)$ on the cardinality of spherical codes with distinct inner products in $[-1, s]$ (see [5]) can be described as follows. For $a, b \in \{0, 1\}$ and $i \geq 1$, let $t_i^{a,b}$ denote the greatest zero of the adjacent Jacobi polynomial $P_i^{(a+\frac{n-3}{2}, b+\frac{n-3}{2})}(t)$. Levenshtein [5] proved that

$$L(n, s) = \begin{cases} L_{2k-1} := \binom{k+n-3}{k-1} \left[\frac{2k+n-3}{n-1} - \frac{P_{k-1}^{(n)}(s) - P_k^{(n)}(s)}{(1-s)P_k^{(n)}(s)} \right], & s \in \left[t_{k-1}^{1,1}, t_k^{1,0} \right] \\ L_{2k} := \binom{k+n-2}{k} \left[\frac{2k+n-1}{n-1} - \frac{(1+s)(P_k^{(n)}(s) - P_{k+1}^{(n)}(s))}{(1-s)(P_k^{(n)}(s) + P_{k+1}^{(n)}(s))} \right], & s \in \left[t_k^{1,0}, t_k^{1,1} \right]. \end{cases} \tag{11}$$

The connection between the Delsarte-Goethals-Seidel bound (10) and the Levenshtein bounds (11) is given by the equalities

$$\begin{aligned} L_{2k-2}(n,t_{k-1}^{1,1}) = L_{2k-1}(n,t_{k-1}^{1,1}) = D(n,2k-1),\\ L_{2k-1}(n,t_{k}^{1,0}) = L_{2k}(n,t_{k}^{1,0}) = D(n,2k). \end{aligned} \tag{12}$$

Levenshtein's method for obtaining his bounds on the cardinality of maximal spherical codes utilizes orthogonal polynomials theory and Gauss-type quadrature rules that we now briefly review.The monotonicity of the bounds $D(n,\tau)$ with respect to τ (see (10)) implies that for every fixed dimension n and cardinality N there is unique $\tau := \tau(n,N)$ such that $N \in (D(n,\tau), D(n,\tau+1)]$.

For the so found τ define $k := \left\lceil \frac{\tau+1}{2} \right\rceil$ and let $\alpha_k = s$ be the unique solution of $N = L_\tau(n,s),\ s \in I_\tau$ (see (12)). Then as described by Levenshtein in [5, Section 5] there exist uniquely determined quadrature nodes and nonnegative weights (we consider odd τ)

$$-1 < \alpha_1 < \cdots < \alpha_k < 1, \quad \rho_1,\dots,\rho_k \in \mathbb{R}^+, \quad i=1,\dots,k \tag{13}$$

such that the Radau $1/N$-quadrature holds

$$f_0 = \frac{f(1)}{N} + \sum_{i=1}^{k} \rho_i f(\alpha_i), \quad \text{for all } f \in \mathcal{P}_\tau. \tag{14}$$

The numbers $\alpha_i,\ i = 1\dots,k$, are the roots of the equation $P_k(t)P_{k-1}(\alpha_k) - P_k(\alpha_k)P_{k-1}(t) = 0$, where $P_i(t) = P_i^{(\frac{n-1}{2},\frac{n-3}{2})}(t)$. In fact, $\{\alpha_i\}$ are roots of the Levenshtein's polynomials $f_\tau^{(n,\alpha_k)}(t)$ (see [5, Equations (5.81) and (5.82)]).

The first ingredient for Theorem 2.1, namely the $1/N$-quadrature rule is given by (14). The optimal polynomials $f(t)$ solving the linear program (4) are Hermite interpolants to the potential at the nodes $\{\alpha_i\}_{i=1}^k$, namely in the notation of Cohn-Kumar [3, p. 110] (over polynomial space $\mathcal{P}_\tau$)

$$f(t) = H(h;(t-s)f_\tau^{(n,s)}(t)), \tag{15}$$

where $f_\tau^{(n,s)}(t)$ are the Levenshtein's extremal polynomials [5].

Theorem 3.1 ([1], Theorem 3.1)**.** *Let n, N be fixed and $h(t)$ be an absolutely monotone potential. Suppose that $\tau = \tau(n,N)$ is as in (**??**), and choose $k = \left\lceil \frac{\tau+1}{2} \right\rceil$. Associate the quadrature nodes and weights α_i and ρ_i, $i = 1,\dots,k$, as in* (14)*. Then*

$$\mathcal{E}(n,N;h) \geq R_\tau(n,N;h) := N^2 \sum_{i=1}^{k} \rho_i h(\alpha_i). \tag{16}$$

Moreover, the polynomials $f(t)$ defined by (15) *provide the unique optimal solution of the linear program* (4) *for the subspace $\Lambda = \mathcal{P}_\tau$, and consequently*

$$\mathcal{W}(n,N,\mathcal{P}_\tau;h) = R_\tau(n,N;h). \tag{17}$$

4 LP-extremal polynomials for $(4, 24)$-codes and improved ULB

The $(4, 24)$-codes take prominence in the literature. In particular, the D_4 root system solving the kissing number problem [6], is suspected to be a maximal code, but is not universally optimal (see [2]). In this case the Levenshtein nodes are $\{-.817352..., -.257597..., .474950...\}$ and the weights are $\{0.138436..., 0.433999..., 0.385897...\}$. Two of the test functions associated with the $1/24$-quadrature rule (14), Q_8 and Q_9, are negative.

Table 1: Test functions for $(4, 24)$-codes, Levenshtein case

Q_6	Q_7	Q_8	Q_9	Q_{10}	Q_{11}	Q_{12}
0.0857	0.1600	−0.0239	−0.0204	0.0642	0.0368	0.0598

Motivated by this we define $\Lambda := \text{span}\{P_0^{(4)}, \ldots, P_5^{(4)}, P_8^{(4)}, P_9^{(4)}\}$. Our main result is a $(4, 24)$-code version of Theorem 2.1.

Theorem 4.1. *The collection of nodes and weights* $\{(\alpha_i, \rho_i)\}_{i=1}^4$

$$\begin{aligned} \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} &= \{-0.86029..., -.0.48984..., -0.19572, .0.478545...\} \\ \{\rho_1, \rho_2, \rho_3, \rho_4\} &= \{0.09960..., 0.14653..., 0.33372..., 0.37847...\}, \end{aligned} \tag{18}$$

define a $1/N$*-quadrature rule that is exact for* Λ*. Moreover, there is a Hermite-type interpolant (see Figure 1)* $H(t) = H(h; (t-\alpha_1)^2 \ldots (t-\alpha_4)^2) \in \Lambda \cap A_{n,h}$*,* $H(\alpha_i) = h(\alpha_i)$*,* $H'(\alpha_i) = h'(\alpha_i)$ *for* $i = 1, \ldots, 4$ *and subsequently the following universal lower bound (and an improvement of* (16)*) holds*

$$\mathcal{E}(n, N; h) \geq N^2 \sum_{i=1}^{4} \rho_i h(\alpha_i). \tag{19}$$

Furthermore, the test functions $Q_j^{(n)}$ *(see* (8)*) are non-negative for all* j*, and therefore* $H(t)$ *is the optimal linear programming solution among all polynomials in* $\mathcal{A}_{n,h}$*.*

The following lemma plays an important role in the proof of the positive definiteness of the Hermite-type interpolants described in Theorem 4.1.

Lemma 4.2. *Suppose* $T := \{t_1 \leq \cdots \leq t_k\} \subset [a, b]$ *is a set of nodes and* $B := \{g_1, \ldots, g_k\}$ *is a linearly independent set of functions on* $[a, b]$ *such that the matrix* $g_B = (g_i(t_j))_{i,j=1}^k$ *is invertible (repetition of points in the multiset*

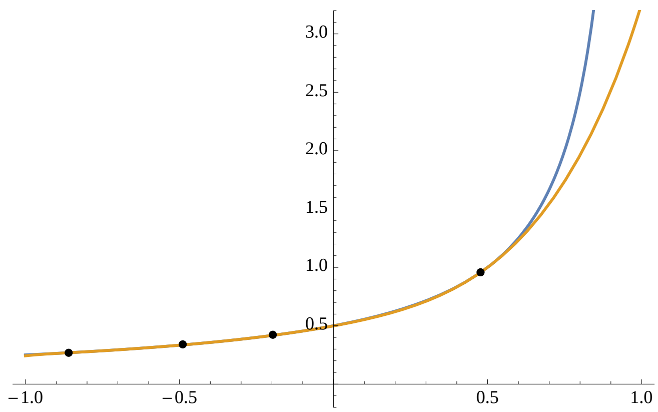

Figure 1: The $(4, 24)$-code optimal interpolant - Coulomb potential

yields corresponding derivatives). Let $H(t, h; \mathrm{span}(B))$ *denote the Hermite-type interpolant associated with* T*. Then*

$$H(t, h; \mathrm{span}(B)) = \sum_{i=1}^{k} h[t_1, \ldots, t_i] H(t, (t - t_1) \cdots (t - t_{i-1}); \mathrm{span}(B)), \quad (20)$$

where $h[t_1, \ldots, t_i]$ *are the divided differences of* h*.*

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