Fifteenth International Workshop on Algebraic and Combinatorial Coding Theory
June 18-24, 2016, Albena, Bulgaria *pp. 78–83*

On maximal antipodal spherical codes with few distances

Peter Boyvalenkov[1] peter@math.bas.bg
Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, 8 G Bonchev Str., 1113 Sofia, Bulgaria
and Faculty of Mathematics and Natural Sciences, South-Western University, Blagoevgrad, Bulgaria.

Konstantin Delchev math_k_delchev@yahoo.com
Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, 8 G Bonchev Str., 1113 Sofia, Bulgaria

Abstract. Using linear programming techniques we derive bounds for antipodal spherical codes. The possibilities for attaining our bounds are investigated and Lloyd-type theorems are proved.

1 Introduction

We are interested in antipodal codes $C \subset \mathbb{S}^{n-1}$ (i.e. $C = -C$) with a few possible distances and maximum possible size provided the dimension and the inner products are fixed. General bounds can be obtained from Levenshtein bound for codes in real projective spaces [8, Section 6], while we consider here the special cases of small number fixed inner products.

If $f(t) \in \mathbb{R}[t]$ is a real polynomial of degree k, then $f(t)$ can be uniquely expanded in terms of the Gegenbauer polynomials as $f(t) = \sum_{i=0}^{k} f_i P_i^{(n)}(t)$. We use the identity (see [5, 8])

$$|C|f(1) + \sum_{x,y \in C, x \neq y} f(\langle x, y \rangle) = |C|^2 f_0 + \sum_{i=1}^{k} f_i M_i \tag{1}$$

as a source of estimations by polynomial techniques. Here

$$M_i := \frac{1}{r_i} \sum_{j=1}^{r_i} \left(\sum_{x \in C} Y_{ij}(x) \right)^2$$

is the i-th moment of C (see [1], [4, Section 2]), the functions $\{Y_{i,j},\ j = 1, 2, \ldots, r_i\}$, are the so-called spherical harmonics of degree i, and $r_i = \binom{n+i-3}{n-2} \frac{2i+n-2}{i}$.

It is clear that C is antipodal if and only if $M_i = 0$ for every odd i. Further, a code C is a spherical τ-design if and only if its moments satisfy $M_i = 0$ for every positive integer $i \leq \tau$ [5, 8].

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[1]The research of this author was supported, in part, by the Bulgarian NSF under contract I01/0003.

This paper is organized as follows. In Section 2 we introduce distance distribution of codes and their derived codes. We also describe the results from [2] in the case of codes with inner products -1 and $\pm s$. Sections 3 and 4 are devoted to the next two steps – we consider codes with inner products -1, 0 and $\pm s$, and -1, $\pm s_1$ and $\pm s_2$, respectively. Upper bounds on the maximal possible size of such codes are derived and investigated as Lloyd type theorems are proved.

2 Some preliminaries

2.1 Distance distributions and derived codes

For fixed $x \in C$, the system of positive integers $(A_t(x) : t \in [-1,1), \exists y \in C, \langle x,y\rangle = t)$, where

$$A_t(x) = |\{y \in C : \langle x,y\rangle = t\}|,$$

is called distance distribution of C with respect to x. The antipodality implies $A_{-1}(x) = 1$ and $A_t(x) = A_{-t}(x)$ for every t and x, and we obviously have $\sum_{t\in(-1,1)} A_t(x) = |C| - 2$. Usually, if a code attains a linear programming bound, then further information about its distance distributions follows. Indeed, the good codes are usually spherical designs of good strength and this forces the distance distributions to satisfy certain equations (see [5, 3]).

For a code $C \subset \mathbb{S}^{n-1}$, a point $x \in C$ and an inner product α, the derived code of C (see [5, Theorem 8.2]) with respect to x and α is the set $C_\alpha(x) = \{y \in C : \langle x,y\rangle = \alpha\}$ re-scaled on $\mathbb{S}^{n-2}$. The inner products of $C_\alpha(x)$ are in the set $I_{\alpha,x} = \left\{\frac{\beta-\alpha^2}{1-\alpha^2} : \beta \text{ is an inner product of } C\right\} \cap [-1,1)$. Moreover, if C is a spherical τ-design then $C_\alpha(x)$ is a spherical $(\tau-\ell+1)$-design, where $\ell = |I_{\alpha,x}|$.

2.2 Antipodal codes with inner products -1 and $\pm s$

Assume that $C \subset \mathbb{S}^{n-1}$ is antipodal, $M = |C|$ and C has inner products -1 and $\pm s$. Such codes are also called (systems of) equiangular lines and are well studied (see, for example [6] and the references therein). It is known (see Theorem 1.2 in [2]) that if $M > 2n$ then $s = \frac{1}{2\ell+1}$, where ℓ is a positive integer. Denote by $M_{2\ell+1}(n)$ the maximum possible size of such C.

Linear programming (and semi-definite programming) bounds for equiangular lines were obtained by Barg and Yu [2] (in slightly different setting), who also give a huge collection of bounds.

Theorem 2.1. *[2] If $P_{2k}^{(n)}(\frac{1}{2\ell+1}) < 0$, then $M_{2\ell+1}(n) \le 2 - \frac{2}{P_{2k}^{(n)}(\frac{1}{2\ell+1})}$.*

Proof. Set $f(t) = P_{2k}^{(n)}(t)$ in (1). Then the RHS equals $M_{2k} \ge 0$. Since $A_s(x) = A_{-s}(x) = \frac{M-2}{2}$ for every $x \in C$, we obtain $2M + M(M-2)P_{2k}^{(n)}(\frac{1}{2\ell+1})$ for the LHS. Therefore $(M-2)P_{2k}^{(n)}(\frac{1}{2\ell+1}) \ge -2$, whence we obtain the desired inequality. □

For $k = 1$ we have $P_2^{(n)}(t) = \frac{nt^2-1}{n-1}$ and therefore $M_{2\ell+1}(n) \le \frac{8n\ell(\ell+1)}{(2\ell+1)^2-n}$ (this is usually called relative bound, see [9]) provided $n < (2\ell+1)^2$.

For $k = 2$ we have $P_4^{(n)}(t) = \frac{(n+2)(n+4)t^4-6(n+2)t^2+3}{n^2-1}$ and therefore

$$M_{2\ell+1}(n) \leq \frac{2(n-2)((2\ell+1)^4(n+2)+6(2\ell+1)^2-n-4)}{6(2\ell+1)^2(n+2)-3(2\ell+1)^4-(n+2)(n+4)} \tag{2}$$

provided $6(2\ell+1)^2(n+2)-3(2\ell+1)^4-(n+2)(n+4) > 0$. The bound (2) is better than the relative bound for $n \geq 96$ and for every ℓ. Of course, this model continues – higher degrees in Theorem 2.1 give better bounds in higher dimensions.

3 Antipodal codes with inner products -1, $\pm s$ and 0

Assume now that $C \subset \mathbb{S}^{n-1}$ of cardinality $M = |C|$ has inner products in -1, $\pm s$ and 0, where $0 < s < 1$. Since C is a spherical 1-design, its distance distribution with respect to any point $x \in C$ satisfies the equation $2A_s(x) + A_0(x) = M - 2$.

The next assertion gives analog to the relative bound for equiangular lines.

Theorem 3.1. *If* $s^2 < \frac{3}{n+2}$, *then*

$$M \leq \frac{2n(n+2)(1-s^2)}{3-s^2(n+2)}. \tag{3}$$

Proof. Using $f(t) = t^2(t^2 - s^2)$ in (1) we get

$$2M(1-s^2) = f_0M^2 + f_2M_2 + f_4M_4, \tag{4}$$

where $f_0 = \frac{3-s^2(n+2)}{n(n+2)}$, $f_2 = \frac{(n-1)(6-s^2(n+4))}{n(n+4)}$ and $f_4 = \frac{n^2-1}{(n+2)(n+4)} > 0$. We have $f_0 > 0 \iff s^2 < \frac{3}{n+2}$, and the last inequality implies $f_2 > 0$ as well. Therefore the RHS of (4) is at most f_0M^2 and we obtain $M \leq \frac{2(1-s^2)}{f_0} = \frac{2n(n+2)(1-s^2)}{3-s^2(n+2)}$. □

If the bound (3) is attained, then $M_2 = M_4 = 0$ which means that C is a spherical 5-design. The 2-design property gives in addition $2s^2A_s(x) = \frac{M}{n} - 2$ for every $x \in C$, whence we easily derive that the distance distributions do not depend on x and

$$A_s(x) = A_s = \frac{M-2n}{2ns^2}, \quad A_0(x) = A_0 = M - 2 - 2A_s = \frac{M(ns^2-1)+n(1-2s^2)}{ns^2}.$$

We consider a derived code of C to obtain a Lloyd-type theorem.

Theorem 3.2. *If C attains the bound* (3) *then s is rational.*

Proof. For fixed $x \in C$, the derived code $C_s(x)$ is a spherical 3-design of cardinality $|C_s(x)| = A_s(x) = \frac{M-2n}{2ns^2}$, inner products $u_1 = \frac{s}{1+s}$, $u_2 = -\frac{s^2}{1-s^2}$ and $u_3 = -\frac{s}{1-s}$. The distance distribution of $C_s(x)$ with respect to $y \in C_s(x)$ satisfies the system (see [5, 3])

$$\begin{aligned} A_{u_1}(y) + A_{u_2}(y) + A_{u_3}(y) &= \frac{M-2n}{2ns^2} - 1 \\ u_1A_{u_1}(y) + u_2A_{u_2}(y) + u_3A_{u_3}(y) &= -1 \\ u_1^2A_{u_1}(y) + u_2^2A_{u_2}(y) + u_3^2A_{u_3}(y) &= \frac{M-2n}{2n(n-1)s^2} - 1 \end{aligned}$$

Simple algebraic manipulations show that the first and second equation imply

$$s(A_{u_1}(y) - A_{u_3}(y)) = \frac{M}{2n} - 2.$$

If s is irrational then $A_{u_1}(y) = A_{u_3}(y)$ and $M = 4n$. This and (3) imply $s^2 = \frac{4-n}{n+2}$ which is possible only for $n = 3$ and leads to the icosahedron which does not have inner product 0. □

When s is rational, the solutions of the above system

$$\left| \begin{array}{lcl} A_{u_1}(y) & = & \frac{(M-4n)(1+s)}{4ns} + \frac{(n-1)M(M^2-8nM+4n^2(n+2))}{4n(3M-2n(n+2))^2} \\ A_{u_2}(y) & = & \frac{(n-1)(M-n(n+1))M^2}{n(3M-2n(n+2))^2} \\ A_{u_3}(y) & = & A_{u_1}(y) - \frac{M-4n}{2ns} \end{array} \right.$$

do not depend on y and must be nonnegative integers.

Another consequence of the known distance distribution (i.e. from the 2-design property) is the analog of Theorem 2.1.

Theorem 3.3. *If C is a spherical 3-design, $k \geq 2$ and $P_{2k}^{(n)}(s) + (ns^2 - 1)P_{2k}^{(n)}(0) < 0$, then*

$$M \leq \frac{n\left(2ns + (1-2s^2)P_{2k}^{(n)}(0) - P_{2k}^{(n)}(s)\right)}{\left|P_{2k}^{(n)}(s) + (ns^2-1)P_{2k}^{(n)}(0)\right|}. \tag{5}$$

Proof. We set $f(t) = P_{2k}^{(n)}(t)$ in (1) and obtain $2M + M(2A_sP_{2k}^{(n)}(s) + A_0P_{2k}^{(n)}(0)) = M_{2k} \geq 0$. Therefore

$$(M-n)P_{2k}^{(n)}(s) + (M(ns^2-1) + n(1-2s^2))P_{2k}^{(n)}(0) \geq -2ns^2,$$

whence the desired inequality follows. □

4 Antipodal codes with inner products -1, $\pm s_1$ and $\pm s_2$

Let $C \subset \mathbb{S}^{n-1}$ of cardinality $M = |C|$ be antipodal and have inner products in -1, $\pm s_1$ and $\pm s_2$, where $0 < s_1 < s_2 < 1$. Then C is a spherical 1-design and its distance distribution with respect to any $x \in C$ satisfies the equation $A_{s_1}(x) + A_{s_2}(x) = \frac{M-2}{2}$ (recall that $A_{-1}(x) = 1$, $A_{-s_1}(x) = A_{s_1}(x)$ and $A_{-s_2}(x) = A_{s_2}(x)$).

Again, we first derive the analog of the relative bound in this case.

Theorem 4.1. *If $s_1^2s_2^2 + \frac{3-(n+2)(s_1^2+s_2^2)}{n(n+2)} > 0$ and $6 - (n+4)(s_1^2 + s_2^2) > 0$, then*

$$M \leq \frac{n(n+2)(1-s_1^2)(1-s_2^2)}{n(n+2)s_1^2s_2^2 - (n+2)(s_1^2+s_2^2) + 3}. \tag{6}$$

Proof. Setting $f(t)=(t^2-s_1^2)(t^2-s_2^2)$ in (1) we obtain.

$$2f(1)M=f_0M^2+f_2M_2+f_4M_4,$$

where $f_0=s_1^2s_2^2+\frac{3-(n+2)(s_1^2+s_2^2)}{n(n+2)}>0$, $f_2=\frac{(n-1)(6-(n+4)(s_1^2+s_2^2))}{n(n+4)}>0$ and $f_4=\frac{n^2-1}{(n+2)(n+4)}>0$. Therefore $M\leq\frac{2(1-s_1^2)(1-s_2^2)}{f_0}$, whence we obtain (6). □

If the bound in Theorem 4.1 is attained, then C must be a spherical 5-design. Thus its distance distributions do not depend on x (so we write $A_{s_1}(x)=A_{s_1}$, $A_{s_2}(x)=A_{s_2}$) and satisfy the equations

$$2(s_1^2A_{s_1}+s_2^2A_{s_2})=\frac{M}{n}-2,\quad 2(s_1^4A_{s_1}+s_2^4A_{s_2})=\frac{3M}{n(n+2)}-2.$$

This implies

$$A_{s_1}=\frac{M-2n-ns_1^2(M-2)}{2n(s_1^2-s_2^2)},\quad A_{s_2}=\frac{M-2n-ns_2^2(M-2)}{2n(s_1^2-s_2^2)}.$$

Theorem 4.2. *If C attains the bound* (6) *then s_1 are simultaneously rational or simultaneously irrational.*

Proof. By calculation of the distance distribution of the derived codes $C_{s_1}(x)$ and $C_{s_2}(x)$. □

We do not expect stronger Lloyd-type theorem here because of the situation with spherical 2-designs which are 2-distance sets [10] where a 13-point spherical 2-design on $\mathbb{S}^5$ is constructed with inner products $\frac{-1\pm\sqrt{13}}{12}$.

Furthermore, the known distance distribution of spherical designs allows analog of Theorems 2.1 and 3.3 from $M_{2k}\geq 0$.

Theorem 4.3. *If C is a spherical 5-design, $k\geq 2$ and $(1-ns_1^2)P_{2k}^{(n)}(s_1)+(1-ns_2^2)P_{2k}^{(n)}(s_2)<0$, then*

$$M\leq\frac{2n\left((1-s_1^2)P_{2k}^{(n)}(s_1)+(1-s_2^2)P_{2k}^{(n)}(s_2)+s_2^2-s_1^2\right)}{\left|(1-ns_1^2)P_{2k}^{(n)}(s_1)+(1-ns_2^2)P_{2k}^{(n)}(s_2)\right|}.\tag{7}$$

Proof. Set $f(t)=P_{2k}^{(n)}(t)$ in (1). □

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