MULTIPLE COVERINGS OF THE FARTHEST-OFF POINTS and MULTIPLE SATURATING SETS in PROJECTIVE SPACES CLASSIFICATION OF MINIMAL 1-SATURATING SETS in PROJECTIVE PLANES

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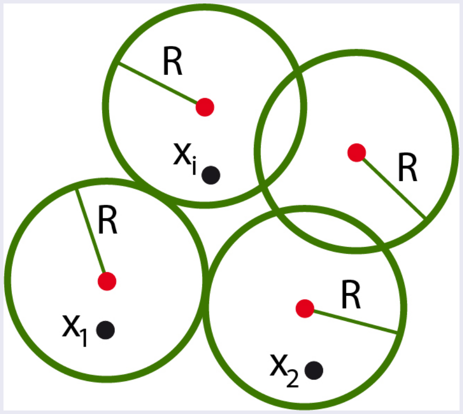
- GENERAL CONSTRUCTIONS of SHORT $(1,\mu)$-SS ($\sim(2,\mu)$-MCF with SMALL 'DENSITY')
- PERFECT $(2,\mu)$-MCF from CLASSICAL GEOMETRICAL OBJECTS

Definition (H. O. Hämäläinen et al., 1995; J. Quistorff, 2001)

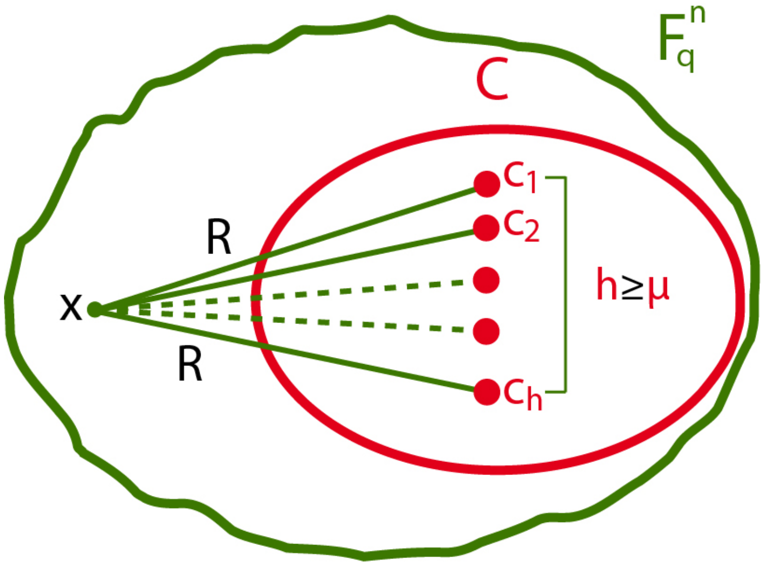
(R, μ)-multiple covering of the farthest-off points ((R, μ)-MCF)

$(n, M, d)_q R$ code $\mathcal{C}$ such that

Length Cardinality Minimum Distance

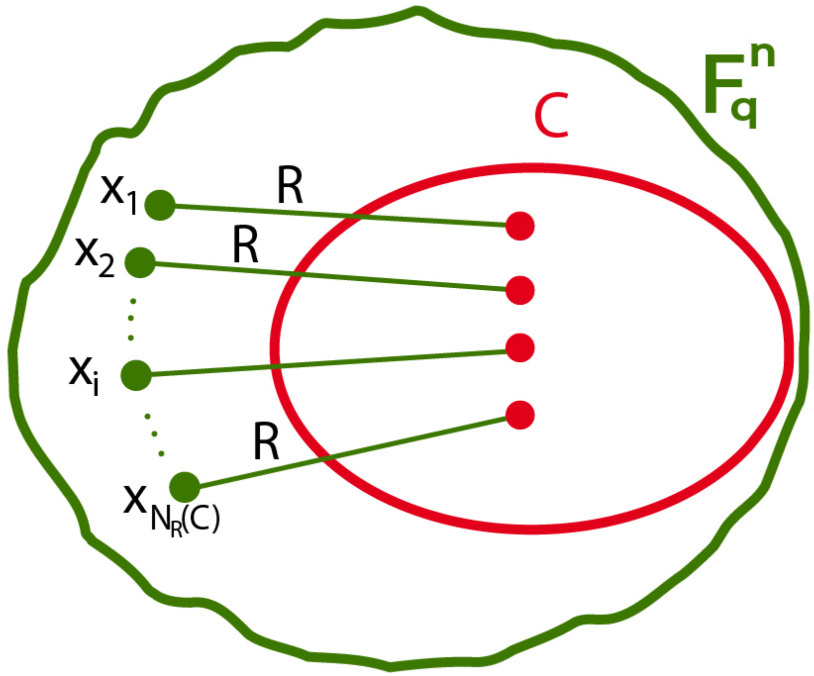


$$\forall x \in \mathbb{F}_q^n : d(x, \mathcal{C}) = R \Longrightarrow |\{c \in \mathcal{C} : d(x, c) = R\}| \geq \mu$$



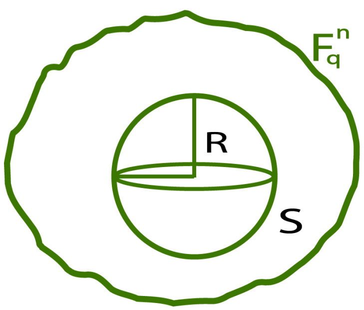
NOTATIONS $\mathcal{C} : (n, M, d(\mathcal{C}))_q R$ code

$$N_R(\mathcal{C}) = |\{x \in \mathbb{F}_q^n : d(x,\mathcal{C}) = R\}|$$



$$M = \#\mathcal{C}$$

$$N_R(\mathcal{C}) \begin{cases} = q^n - MV_q(n, R-1) & \text{if } d(\mathcal{C}) \geq 2R-1 \\ > q^n - MV_q(n, R-1) & \text{if } d(\mathcal{C}) < 2R-1 \end{cases}$$



$$|S| = V_q(n, R) = \sum_{i=0}^{R} \binom{n}{i} (q-1)^i$$

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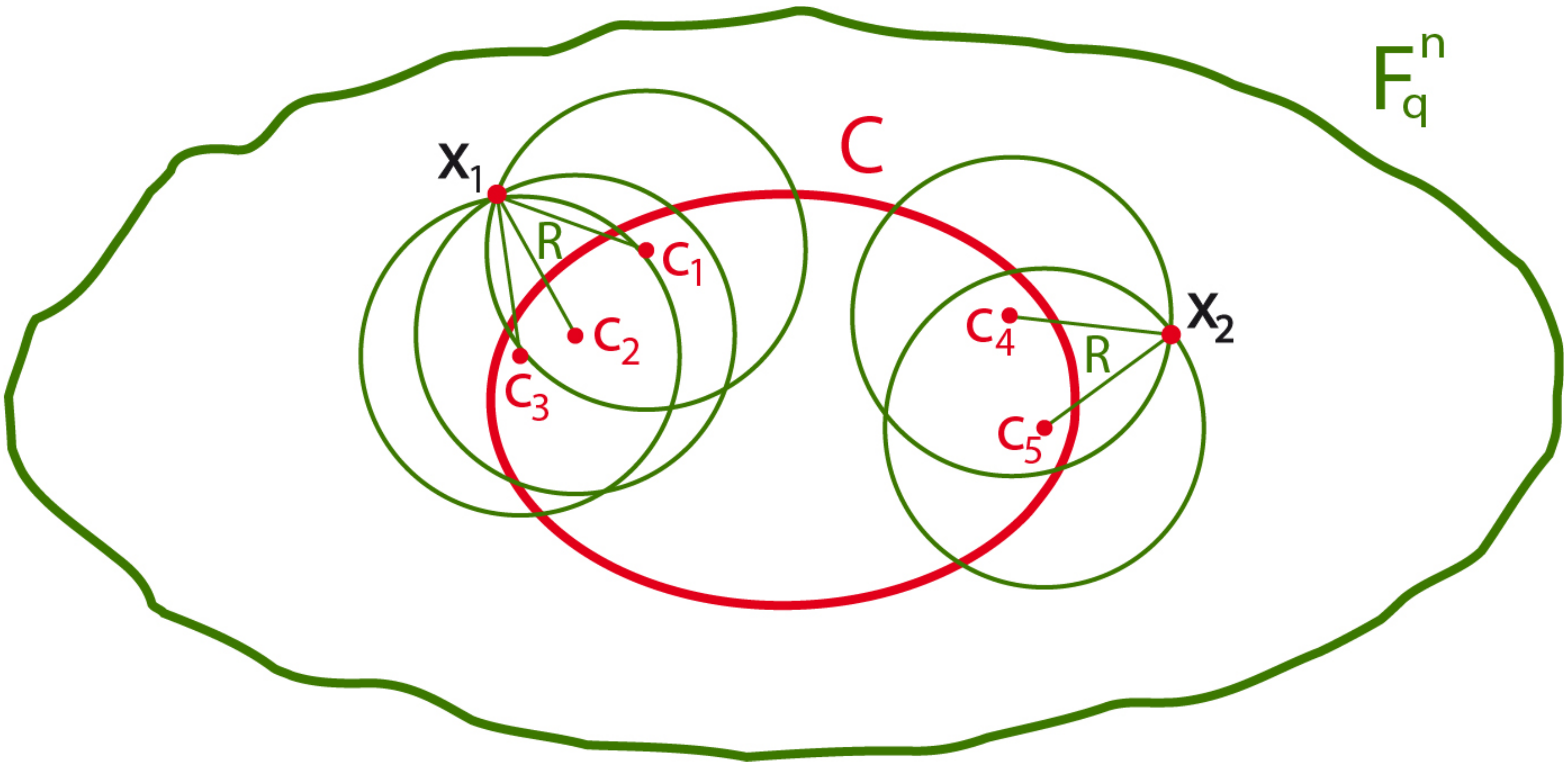

AVERAGE NUMBER OF SPHERES $\gamma(\mathcal{C},R)\le \dfrac{M\binom{n}{R}(q-1)^R}{N_R(\mathcal{C})}$

- of radius R
- center $c\in\mathcal{C}$
- containing $x\in\mathbb{F}_q^n$, $d(x,c)=R$

μ-DENSITY

Definition

Let an $(n, M, d)_q R$ code $\mathcal{C}$ be (R, μ)-MCF

$$\mu\text{-}\mathbf{density}\ \delta_\mu(\mathcal{C}, R) := \frac{\gamma(\mathcal{C}, R)}{\mu} \leq \frac{M\binom{n}{R}(q-1)^R}{N_R(\mathcal{C})}$$

$$\textbf{Perfect } (R, \mu)\text{-MCF} : \delta_\mu(\mathcal{C}, R) = 1$$

Remark

If $\mathcal{C}$ linear $[n, k, d(\mathcal{C})]_q R$-code, $d(\mathcal{C}) \geq 2R - 1$,

$$\delta_\mu(\mathcal{C}, R) \leq \frac{\binom{n}{R}(q-1)^R}{\mu(q^{n-k} - V_q(n, R-1))}$$

SMALL μ-density $\Longrightarrow$ **BETTER (R, μ)-MCF codes**

MULTIPLE SATURATING SETS

Definition

$\mathcal{I} \subset PG(N,q)$ (ρ,μ)-**saturating set**, $1 \le \rho \le N$, $\mu \ge 1$, if

1. $\langle \mathcal{I} \rangle = PG(N,q)$
2. $\exists\, Q \in PG(N,q) \,:\, Q \notin \langle P_{i_1},\dots,P_{i_h}\rangle = \mathcal{U}$, $\dim \mathcal{U} = \rho - 1$ $\{P_{i_1},\dots,P_{i_h}\} \subset \mathcal{I}$
3. $|\{T = \langle P_{i_1},\dots,P_{i_k}\rangle : P_{i_j} \in \mathcal{I}, \dim(T) = \rho \text{ and } Q \in T\}| \ge \mu$

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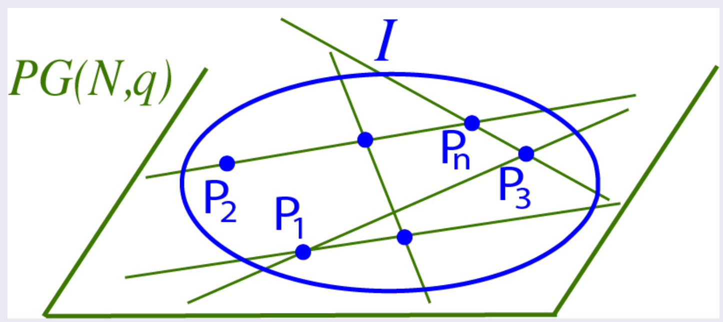

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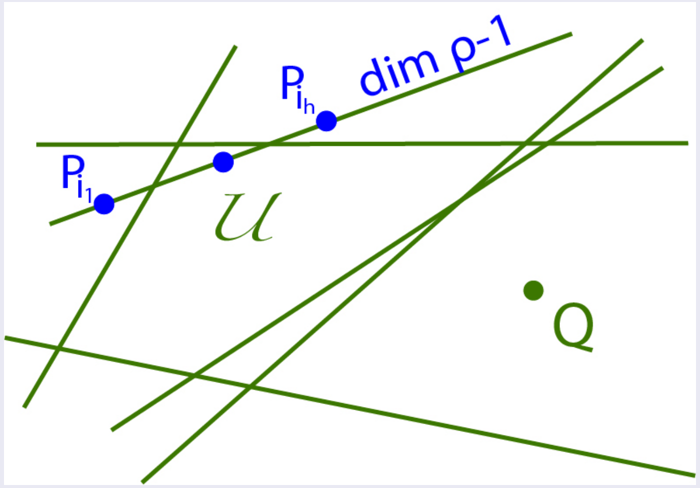

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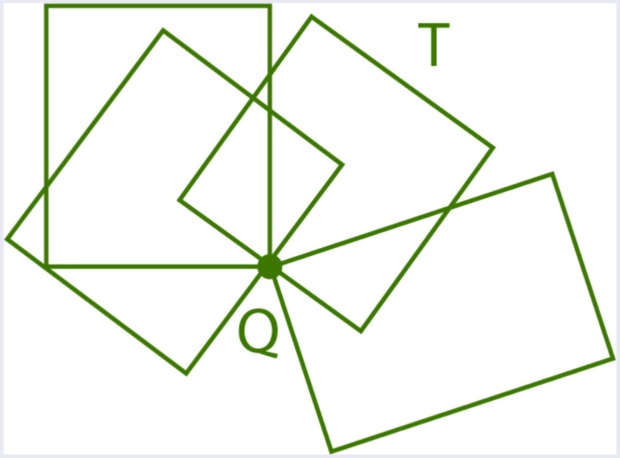
Counted with multiplicity m_T



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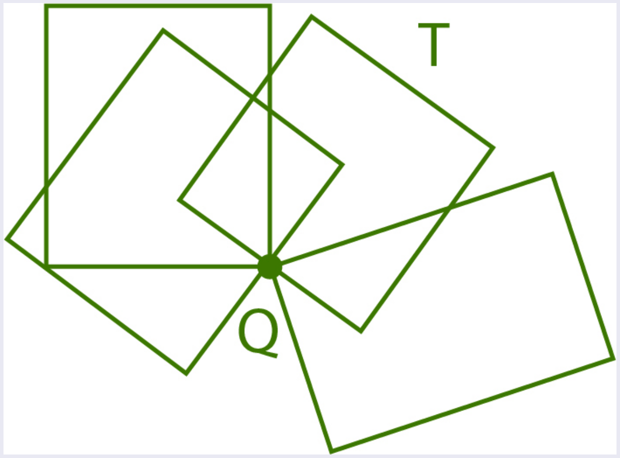


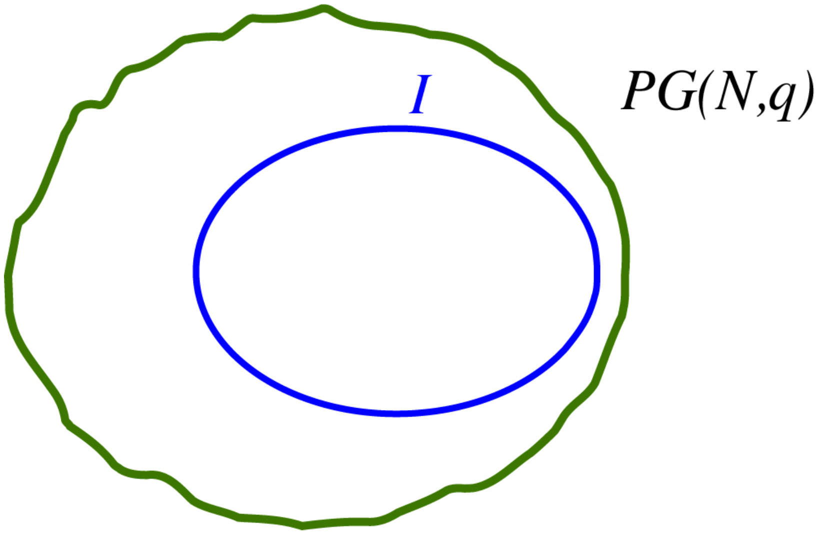
Definition

$\mathcal{I}$ (ρ,μ)-saturating set is **minimal** if $\mathcal{I} \not\subset \mathcal{I}'(\rho,\mu)$-saturating set

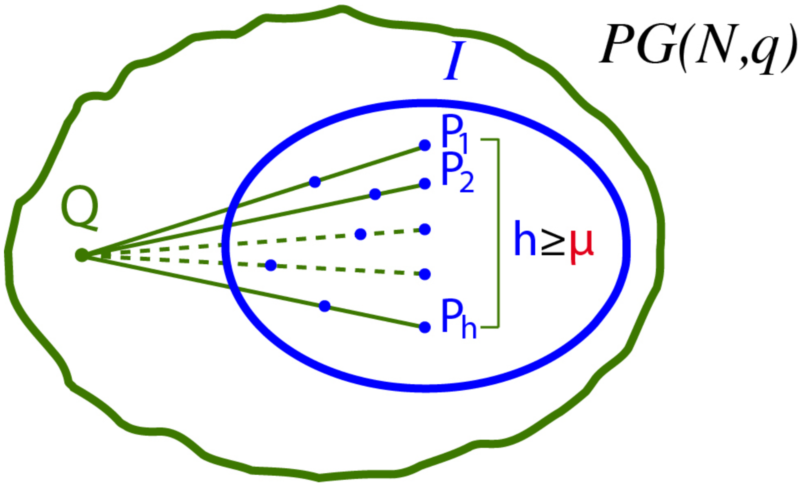
SPECIAL CASES

$\boxed{\rho = 1}$ $\mathcal{I} \subset PG(N,q)$ $(1,\mu)$-saturating set

M2: $\mathcal{I} \neq PG(N,q)$



M3:

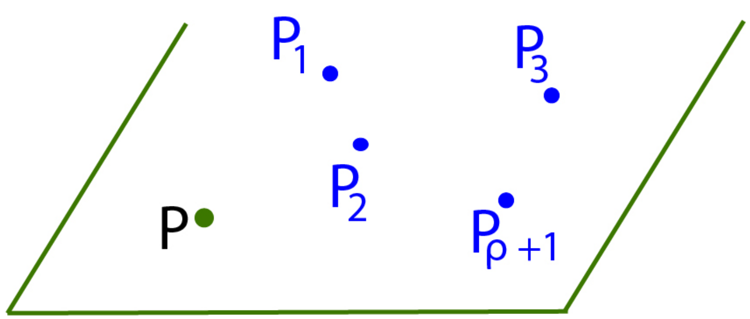


SPECIAL CASES

$\boxed{\mu = 1}$ $\mathcal{I}$ $(\rho, 1)$-saturating set in $PG(N, q) \to$ $\boxed{\rho\text{- saturating set}}$

ρ is the smallest integer :

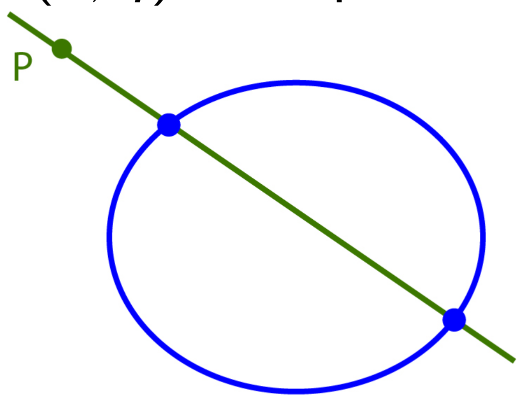
$$\forall\ P \in PG(N,q)\ \exists \{P_1, P_2, \ldots, P_{\rho+1}\} \subset \mathcal{I} : P \in \langle P_1, P_2, \ldots, P_{\rho+1} \rangle$$



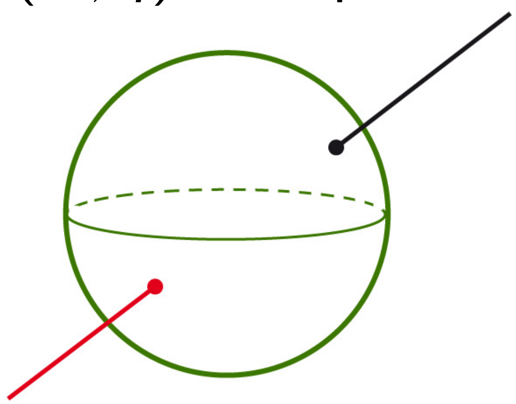
Examples 1 - saturating sets

Minimal

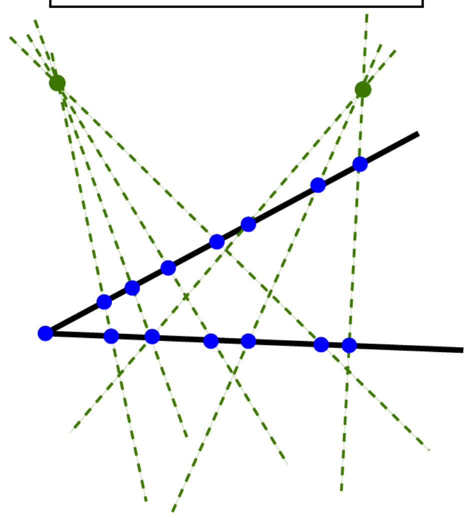
$PG(2, q)$: complete arcs



$PG(N, q)$: complete caps



Non-minimal



(ρ,μ)-SATURATING SETS and CODING THEORY

$\{P_1, \quad P_2, \quad \ldots, \quad P_n\}$ (ρ,μ)-saturating set in $PG(n-k-1,q)$

$\downarrow \quad \downarrow \quad \downarrow$

$(H^1, \quad H^2, \quad \ldots \quad H^n)$ parity check matrix of a linear $[n,k]_q R$-code $\mathcal{C}$
$R = \rho + 1$

$\Updownarrow$ *Lemma*

$\mathcal{C}$ is a linear $(\rho+1,\mu)$-MCF code

(ρ,μ)-SATURATING SETS $\longleftrightarrow$ LINEAR $(\rho+1,\mu)$-MCF CODES

$\rho = 1$ $(R = 2)$ $\quad d(\mathcal{C}) \geq 2R - 1 = 3 \sim$ no multisets

$\mathcal{C}$ $\quad [n, n-(N+1), d(\mathcal{C})]_q 2$ code, $d(\mathcal{C}) \geq 3$ and $(2,\mu)$-MCF

($\sim$ $(1,\mu)$-saturating set of size n in $PG(N,q)$)

$\Downarrow$

μ − **density** $\qquad \delta_\mu(\mathcal{C}, n) \leq \dfrac{\frac{1}{2}(n-1)(q-1)}{\mu\left(\frac{\#PG(N,q)}{n} - 1\right)}$

Fixed q, N, μ, SMALL n $\Longrightarrow$ SMALL μ-**density**

Definition

$\ell_\mu(2, N+1, q)$: **smallest length** n

$\delta_\mu(2, N+1, q)$: **minimum** μ**-density**

$\rho = 1$ $(R = 2)$ $d(\mathcal{C}) \geq 2R - 1 = 3 \sim$ no multisets

$\ell_\mu(2, N+1, q)$: **smallest length** n

$\delta_\mu(2, N+1, q)$: **minimum** μ**-density**

Remark

$$\mu = 1 \Longrightarrow \ell_\mu(2, N+1, q) = \ell(2, N+1, q)$$

↑ *smallest length of 1-saturating sets in* $PG(N, q)$

$$\ell_\mu(2, N+1, q) \leq \mu\ell(2, N+1, q)$$

$$\delta_\mu(2, N+1, q) \leq \frac{\frac{1}{2}(\mu\ell(2,N+1,q)-1)(q-1)}{\mu\left(\frac{\#PG(N,q)}{\mu\ell(2,N+1,q)}-1\right)} \sim \mu\delta(2, N+1, q)$$

↑ *smallest density of 1-saturating sets in* $PG(N, q)$

$\rho = 1$ $(R = 2)$ $d(\mathcal{C}) \geq 2R - 1 = 3 \sim$ no multisets

Principal Aim

Construct linear $(2, \mu)$-MCF codes with
μ-density $\delta_\mu < \mu\delta(2, N+1, q)$ (or $< \mu\bar{\delta}(2, N+1, q)$)

↑ smallest known

$\sim$ equivalent to

Construct $(1, \mu)$-saturating sets in $PG(N, q)$ with
size $n < \mu\ell(2, N+1, q)$ (or $< \mu\bar{\ell}(2, N+1, q)$)

↑ smallest known

$\rho = 1$ $(R = 2)$ $\quad d(\mathcal{C}) \geq 2R - 1 = 3 \sim$ no multisets

$\exists$ SMALL 1-SATURATING SETS in $PG(N,q)$ $\Longrightarrow$ ESTIMATES for $\ell_\mu(2, N+1, q)$ and $\delta_\mu(2, N+1, q)$

$$\bar{\ell}_1(2, N+1, q) = \bar{\ell}(2, N+1, q) = \ ?$$

Best known general result (E. Boros, T. Szőnyi, K. Tichler, *DM* 2005)

in $PG(2,q)$ $\exists$ 1-saturating set of size $\underbrace{\lfloor 5\sqrt{q \log q}\rfloor}_{\bar{\ell}(2,3,q)}$ $\leftarrow$ Probabilistic methods

$\Downarrow$

$$\ell_\mu(2,3,q) \leq \mu\lfloor 5\sqrt{q\log q}\rfloor$$

$\Downarrow$

$$\delta_\mu(2,3,q) \leq \frac{\frac{1}{2}(\mu\lfloor 5\sqrt{q\log q}\rfloor - 1)(q-1)}{\mu\left(\frac{q^2+q+1}{\mu\lfloor 5\sqrt{q\log q}\rfloor} - 1\right)}$$

Classification of minimal 1-saturating sets in $PG(2,q)$

Complete classification $q \le 11$:

- $q \le 8$: S. Marcugini, F. P., Australasian J. of Comb. 2003
- $q = 9, 11$: ACCT2012

$PG(2,9)$	$k=6$	G_{120}: **1**			
	$k=7$	$\mathbb{Z}_4$: 1	G_{42}: **1**	G_{120}: 1	
	$k=8$	$\mathbb{Z}_1$: 88	$\mathbb{Z}_2$: 52	$\mathbb{Z}_2\times\mathbb{Z}_2$: 11	$\mathcal{S}_3$: 1
		$\mathbb{Z}_2\times\mathbb{Z}_4$: 1	$\mathcal{D}_4$: 1	$\mathcal{D}_6$: 3	G_{16}: 1+**1**
		G_{24}: 2	G_{48}: 1		
	$k=9$	$\mathbb{Z}_1$: 667	$\mathbb{Z}_2$: 87	$\mathbb{Z}_3$:9	$\mathbb{Z}_2\times\mathbb{Z}_2$: 4
		$\mathcal{S}_3$: 2	$\mathcal{D}_4$: 1	$\mathcal{D}_6$: 1	G_{16}: 1
		G_{48}: 1			
	$k=10$	$\mathbb{Z}_1$: 58	$\mathbb{Z}_2$: 22	$\mathbb{Z}_4$:5	$\mathbb{Z}_2\times\mathbb{Z}_2$: 4
		$\mathcal{D}_4$: 2	G_{16}: 1	G_{20}: 1	G_{32}: 1+**1**
		G_{1440}: 1			
	$k=11$	G_{11520}: 1			
$PG(2,11)$	$k=7$	$\mathbb{Z}_7\rtimes\mathbb{Z}_3$: **1**			
	$k=8$	$\mathbb{Z}_1$: 22	$\mathbb{Z}_2$: 26+**5**	$\mathbb{Z}_2\times\mathbb{Z}_2$: 2+**1**	$\mathcal{D}_4$: 1 + **1**
		$\mathcal{D}_5$: **1**	G_{16}: **1**		
	$k=9$	$\mathbb{Z}_1$: 10686	$\mathbb{Z}_2$: 265+**1**	$\mathbb{Z}_3$: 40 +**1**	$\mathbb{Z}_4$: 2
		$\mathbb{Z}_2\times\mathbb{Z}_2$: 3	$\mathcal{S}_3$: 10+**1**	$\mathbb{Z}_{10}$: 1	$\mathcal{Q}_6$: 1
	$k=10$	$\mathbb{Z}_1$: 115731	$\mathbb{Z}_2$: 1332	$\mathbb{Z}_3$: 31	$\mathbb{Z}_4$: 15
		$\mathbb{Z}_2\times\mathbb{Z}_2$: 13	$\mathbb{Z}_5$: 2	$\mathcal{S}_3$: 8	$\mathcal{D}_4$: 2
			$\mathcal{D}_5$: 2	$\mathbb{Z}_{10}$: 1$\mathcal{Q}_6$:1	G_{60}:**1**
	$k=11$	$\mathbb{Z}_1$: 30802	$\mathbb{Z}_2$: 147	$\mathbb{Z}_4$: 1	$\mathbb{Z}_2\times\mathbb{Z}_2$: 3
		$\mathcal{D}_4$: 3			
	$k=12$	$\mathbb{Z}_1$: 119	$\mathbb{Z}_2$: 7	$\mathbb{Z}_3$: 5	$\mathcal{S}_3$: 1
		$\mathcal{Q}_6$: 1	G_{20}: 1	G_{1320}: **1**	
	$k=13$	G_{13200}: 1			

Classification of minimal 1-saturating sets in $PG(2,q)$

Classification of the smallest size $13 \le q \le 23$:

- $q \le 13$: S. Marcugini, F. P., Australasian J. of Comb. 2003
- $q = 16, 17, 19, 23$: ACCT2012

$PG(2,16)$	$k=9$	$\mathbb{Z}_3$: **1**	$\mathbb{Z}_6$: **1**	$\mathcal{D}_6$: 1	$\mathcal{G}_{54}$: 1
	$k=10$	$\mathbb{Z}_1$: 7744+**342**	$\mathbb{Z}_2$: 699+**130**	$\mathbb{Z}_3$: 3	$\mathbb{Z}_4$: 12+**8**
		$\mathbb{Z}_2 \times \mathbb{Z}_2$: 27+**4**	$\mathbb{Z}_6$: 2	$\mathcal{S}_3$: 4+**3**	$\mathcal{D}_4$: 8
		$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$: 18+**10**	$\mathcal{Q}_6$: 1	$\mathbb{Z}_4 \times \mathbb{Z}_4$: 1	G_{16}: 4+**3**
		G_{20}: **1**	G_{24}: 1	G_{32}: 1	G_{48}: 1
$PG(2,17)$	$k=10$	$\mathbb{Z}_1$: 2591+**341**	$\mathbb{Z}_2$: 460+**179**	$\mathbb{Z}_3$: 8+**10**	$\mathbb{Z}_4$: 4+**7**
		$\mathbb{Z}_2 \times \mathbb{Z}_2$: 5+**8**	$\mathcal{S}_3$: 7+**9**	$\mathcal{D}_4$: 4	$\mathcal{Q}_4$: **1**
		$\mathcal{Q}_6$: **2**	G_{16}: 1+**1**	G_{18}: **1**	G_{24}: **1**
$PG(2,19)$	$k=10$	$\mathbb{Z}_1$: 1+**1**	$\mathbb{Z}_2$: 6+**18**	$\mathbb{Z}_3$: **1**	$\mathbb{Z}_4$: **1**
		$\mathbb{Z}_2 \times \mathbb{Z}_2$: **2**	$\mathcal{S}_3$: **2**	$\mathcal{D}_5$: **2**	$\mathcal{Q}_6$: **1**
		G_{60}: **1**			
$PG(2,23)$	$k=10$	$\mathcal{S}_3$: **1**			

- $25 \le q \le 1217$ $\bar{\ell}(2,3,q)$: A.A. Davydov, M. Giulietti, S. Marcugini, F. P., Adv. Math. Commun. 2011

MINIMAL $(1,\mu)$-SATURATING SETS in $PG(2,q)$ ($\mu \leq \frac{q(q^2-1)}{2}$)

Remark

Lower bound

$$\ell_{\mu}(2,3,q) \geq \sqrt{2\mu q}$$

Definition

$$m_{\mu}(2,3,q)$$

MAXIMUM size of a minimal $(1,\mu)$-saturating set in $PG(2,q)$

Theorem (MAXIMUM SIZE)

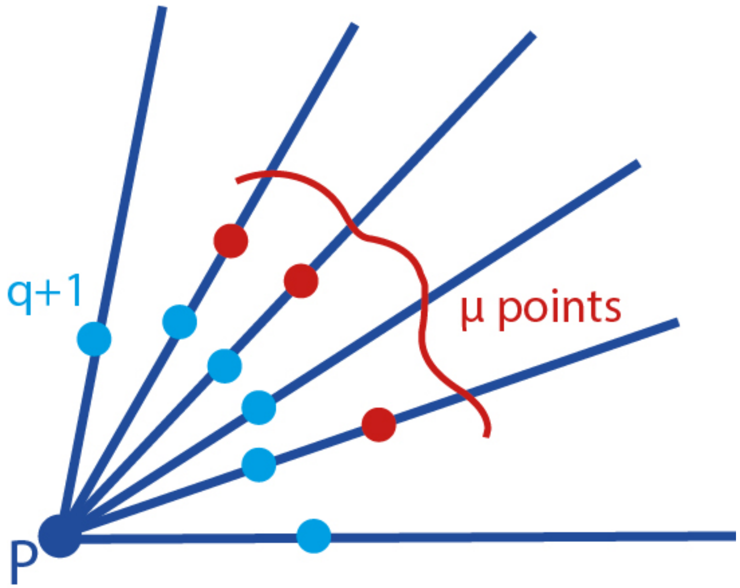
In $PG(2,q)$, $q>2$

$$m_{\mu}(2,3,q) \leq q+\mu+1$$

$$\boxed{\mu \leq q \Longrightarrow m_{\mu}(2,3,q) = q+\mu+1}$$

Proof (Theorem maximum size)

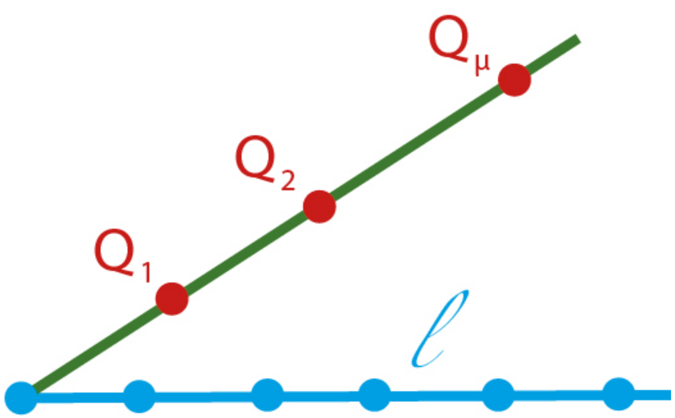
$$|\mathcal{K}| = q + \mu + 1$$



$\Longrightarrow \mathcal{K}$ is $(1,\mu)$-saturating

minimal?

<u>Example</u> : $\mu \leq q$



$$\ell \cup \{Q_1, \ldots, Q_\mu\} = \mathcal{K}$$

<u>minimal</u>

MINIMAL (1, 2)-SATURATING SETS in $PG(2, q)$

SPECTRUM : $\forall\, q$ SEVERAL SIZES $< 2\ell(2,3,q)$ (or $2\bar{\ell}(2,3,q)$)

COMPLETE CLASSIFICATIONS $3 \leq q \leq 9$

q	$\ell(2,3,q)$	$\ell_2(2,3,q)$	Spectrum
2	4^1	5	$5^1 6^1$
3	4^1	6	6^4
4	5^1	6	$6^2 7^5$
5	6^6	6	$6^1 7^4 8^{18}$
7	6^3	8	$8^{13} 9^{564} 10^{424}$
8	6^1	8	$8^2 9^{154} 10^{3372} 11^{611}$
9	6^1	8	$8^1 9^{57} 10^{12145} 11^{76749} 12^{3049}$

MINIMAL $(1,2)$-SATURATING SETS in $PG(2,q)$

COMPLETE SPECTRUM and PARTIAL CLASSIFICATION $11 \leq q \leq 17$

q	$\bar{\ell}(2,3,q)$	$\ell_2(2,3,q)$	Spectrum
11	7^1	10	$10^{1348}[11-13,14]$
13	8^2	10	$10^2 11^{50794}[12-15,16]$
16	9^4	11	$11^{52}[12-17,18,19]$
17	10^{3640}	12	$[12-19,20]$

MINIMAL (1, 2)-SATURATING SETS in $PG(2, q)$

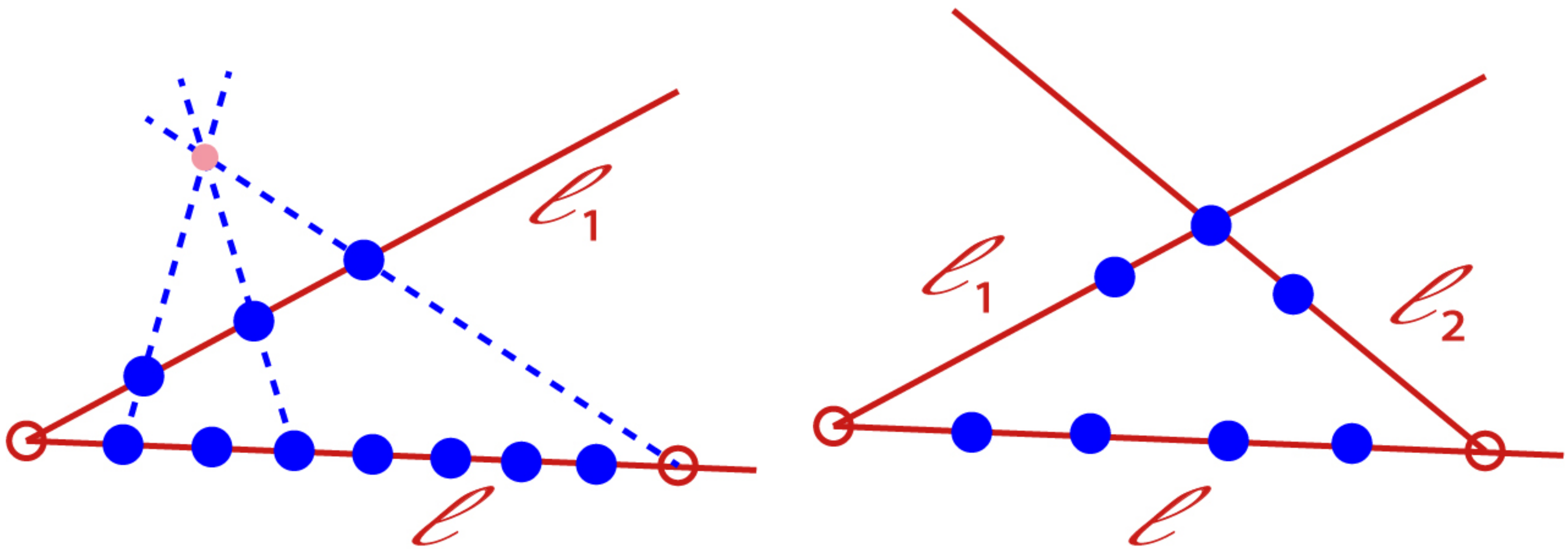
SIZES OF SPECTRUM $19 \leq q \leq 49$

q	$\bar{\ell}(2,3,q)$	Trivial lower bound for $\ell_2(2,3,q)$: $2\sqrt{q}$	Found sizes
19	10^{36}.	9	[13 − 19, 20 − 22]
23	10^{1}.	10	[15 − 19, 20 − 26]
25	12	10	[17 − 23, 24 − 28]
27	12	11	[17 − 23, 24 − 30]
29	13	11	[19 − 25, 26 − 32]
31	14	12	[19, 21 − 27, 28 − 34]
32	13	12	[20 − 25, 26 − 35]
37	15	13	[23, 26 − 29, 30 − 40]
41	16	13	[25, 29 − 31, 32 − 44]
43	16	14	[25, 30, 31, 32 − 46]
47	18	14	[27, 34, 35, 36 − 50]
49	18	14	[29, 34, 35, 36 − 52]

GEOMETRIC CONSTRUCTIONS

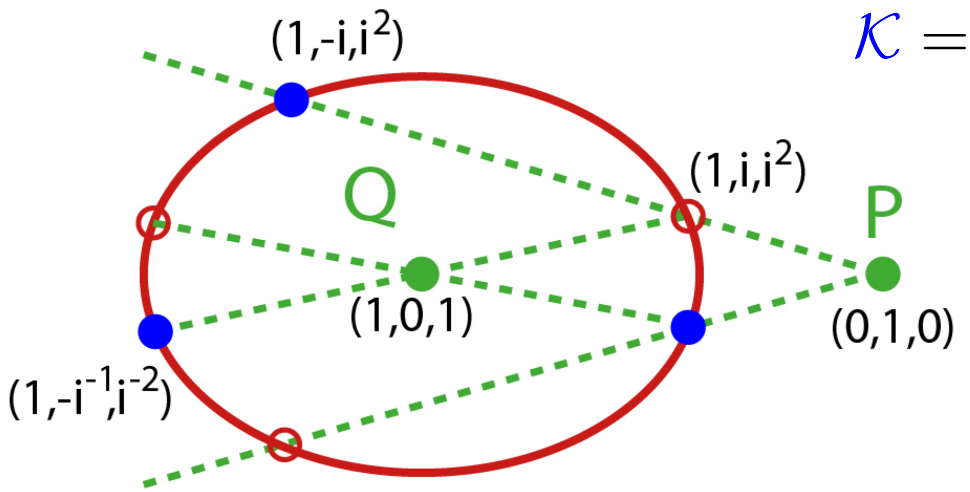
MOST MEANINGFUL SIZES OBTAINED BY CONSTRUCTIONS

Constructions of minimal $(1, 2)$-saturating sets of size $q + 2$



GEOMETRIC CONSTRUCTIONS

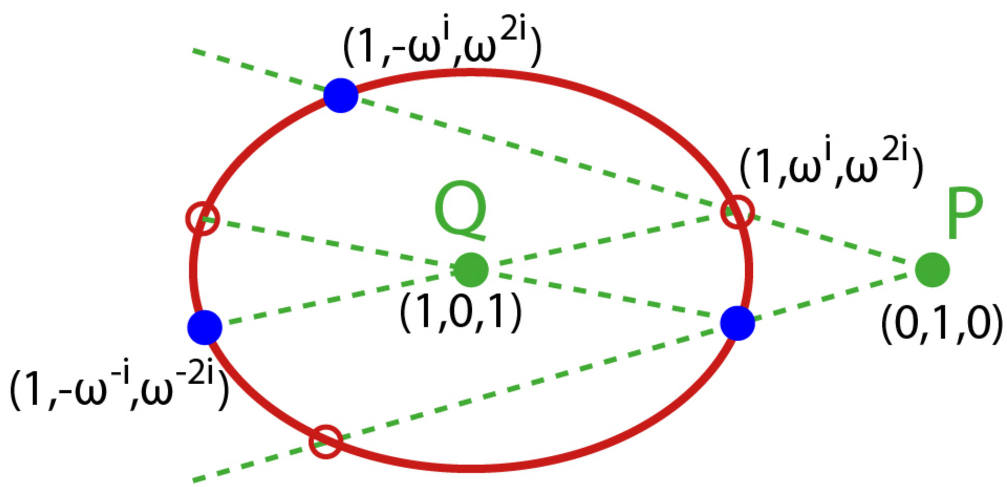
MOST MEANINGFUL SIZES OBTAINED BY CONSTRUCTIONS



$$\mathcal{K} = \text{Conic} \setminus \{(0,i,i^2) | i \in \square\} \cup \{P,Q\}$$

$$q \equiv 3 \bmod 4$$

$$|\mathcal{K}| = \frac{q+9}{2}$$



$$q \equiv 1 \bmod 4$$

$$|\mathcal{K}| = \frac{q+7}{2}$$

Construction of small $(1, \mu)$-saturating sets in $PG(2, q)$

$$\mathbb{F}_q\text{: } q = p^\ell,\ p \text{ prime},\ H < \mathbb{F}_q,\ |H| = p^s,\ 2s < \ell$$

additive

$$L_H(x) = \prod_{h \in H} (x - h) \in \mathbb{F}_q[x] \Longrightarrow L_H(x) = \sum_{i=0}^{s} \beta_i X^{p^i}$$

linearized polynomial

$$M_H = \left\{ \left(\frac{L_{H_1}(\beta_1)}{L_{H_2}(\beta_2)} \right)^p \,|\, H_1, H_2 < H, |H_1| = |H_2| = p^{s-1}, \beta_i \in H \setminus H_i \right\}$$

$\boxed{\gamma \in \mathbb{F}_q}$ $\quad D_{H,\gamma} = \{(L_H(a) : \gamma : 1) \mid a \in \mathbb{F}_q\} \subset PG(2, q)$

Theorem

$q = p^\ell$, ℓ *odd*, $1 \leq \mu \leq p$, $\ell = 2s + 1$, $H < \mathbb{F}_q$, $|H| = p^s$

$\forall\ v \geq 1\ \exists\ (1, \mu)$*-saturating set* $T \subset PG(2, q)$

$$\boxed{\#T \leq (v + 1)p^{s+1} + \mu \frac{|M_H|^v}{(q-1)^{v-1}} + 1 + \mu}$$

Corollary

$q = p^{2s+1}$, $1 \le \mu \le p$

$$\exists\ (1,\mu)\text{-}\textit{saturating set } T \subset PG(2,q)$$

$$\#T \le n_\mu(s,p,v)$$

$$||$$

$$\min_{v=1,\dots,2s+1} \left\{ (v+1)p^{s+1} + \mu \frac{(p^s-1)^{2v}}{(p-1)^v (p^{2s+1}-1)^{v-1}} + 1 + \mu \right\}$$

$\exists$ triples (s,p,v) (On small dense sets in Galois planes, Table 1 M. Giulietti Electr. J. Comb. 2007) :

$$n_1(s,p,v) = \bar{\ell}(2,3,q) \quad < \quad \left\lfloor 5\sqrt{q \log q} \right\rfloor$$

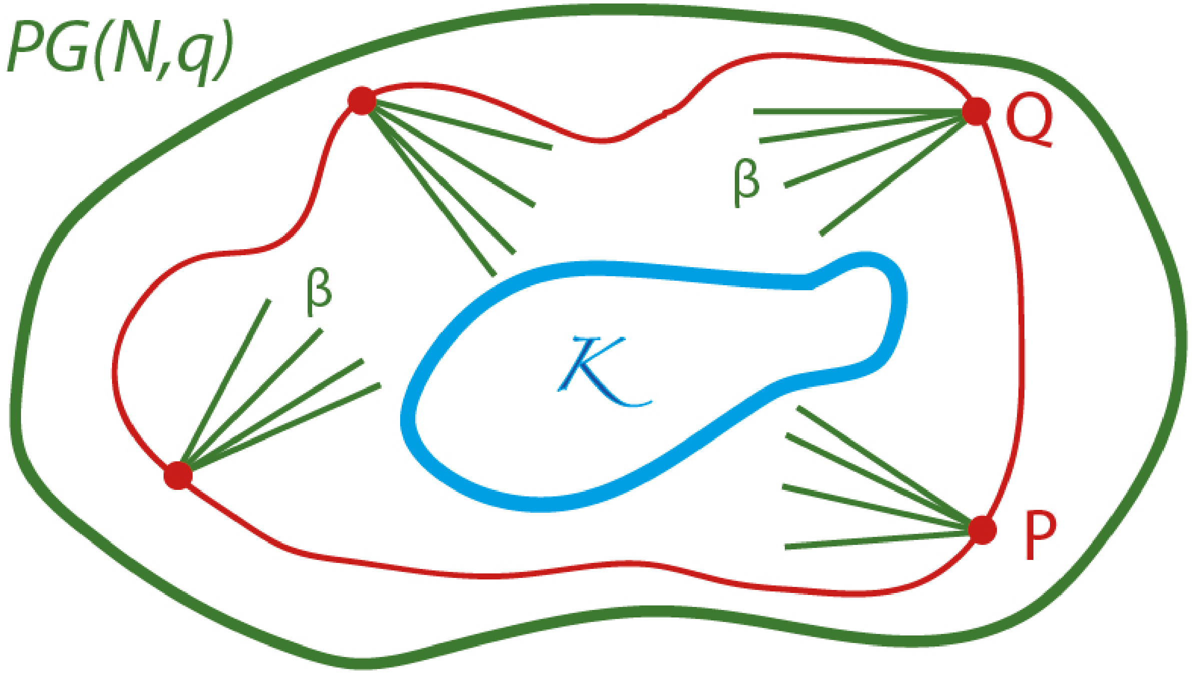
(for the corresponding q)

$$\Downarrow$$

$$\mu \ge 2$$

$$n_\mu(s,p,v) < \mu n_1(s,p,v) < \mu \left\lfloor 5\sqrt{q \log q} \right\rfloor$$

SHORT $(2,\mu)$-MCF CODES FROM CLASSICAL GEOMETRICAL OBJECTS



$P \notin \mathcal{K}$

$\forall Q \in G(P)$, G stabilizer group of $\mathcal{K}$

$$\left| \begin{array}{c} \text{bisecants} \\ \text{to } \mathcal{K} \text{ through } Q \end{array} \right| = \beta \ (\underline{\text{constant}})$$

If $G(P) \equiv PG(N,q) \setminus \mathcal{K} \Longrightarrow \mu = \beta \qquad \forall P \in PG(N,q) \setminus \mathcal{K}$

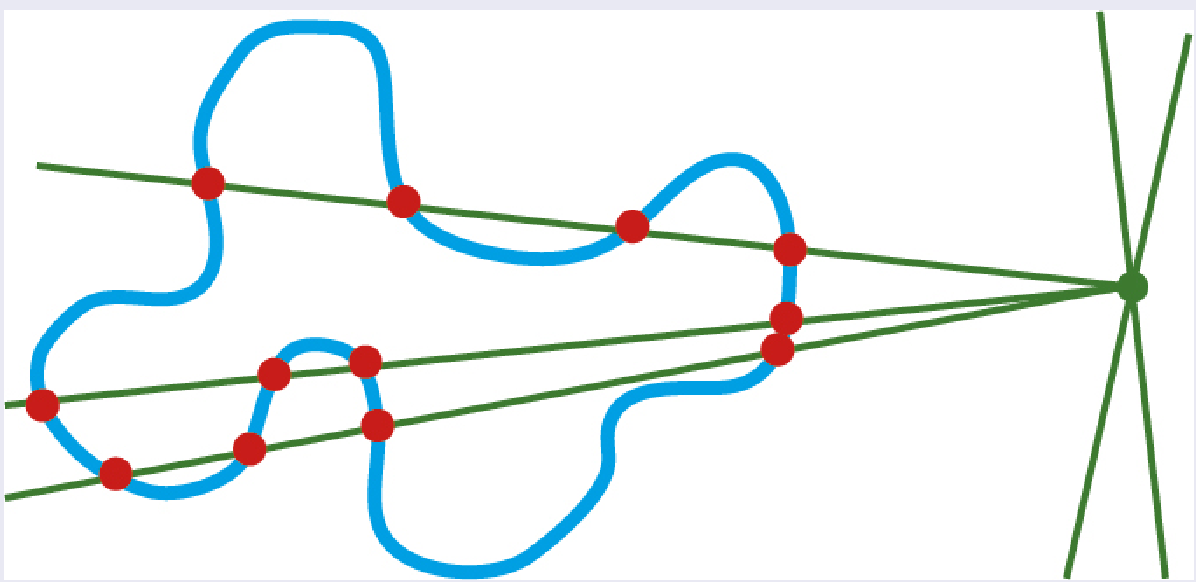
Theorem (MAXIMAL (n, s)-arcs)

$$q = 2^v \quad s = 2^k \quad 1 \le k \le v \quad n = (s-1)q + s$$

$$\text{MAXIMAL } (n, s)\text{-arcs } \mathcal{K} \text{ in } PG(2, q) \longleftrightarrow \mathcal{C} : [n, n-3]_q 2 \text{ code}$$

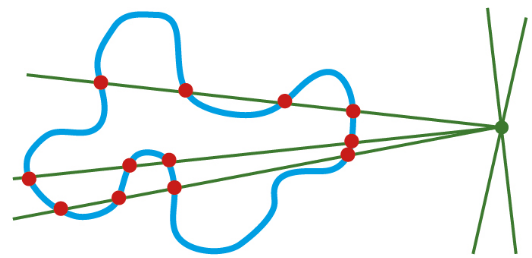
$$\mathcal{C} \text{ is } \Big(\underbrace{2}_{R}, \underbrace{\frac{n}{2}(s-1)}_{\mu}\Big) MCF$$

$$\delta_\mu(\mathcal{C}, 2) = 1 \qquad \underline{PERFECT}$$



proof (sketch)

$$\mathcal{K} \text{ maximal } (n, s) \text{ arc} \Longrightarrow \forall \text{ line } \ell \quad |\ell \cap \mathcal{K}| = \begin{cases} 0 \\ \text{or} \\ s \end{cases}$$



$$\forall P \notin \mathcal{K} \quad |s\text{-secants through } P| = \frac{n}{s}$$

$$\Longrightarrow \mu = \binom{s}{2}\frac{n}{s} = \frac{n(s-1)}{2} \Longrightarrow \boxed{\mathcal{C} \text{ is } \Big(\underbrace{2}_{R}, \underbrace{\tfrac{n}{2}(s-1)}_{\mu}\Big)\text{MCF}}$$

↑ Bisecants

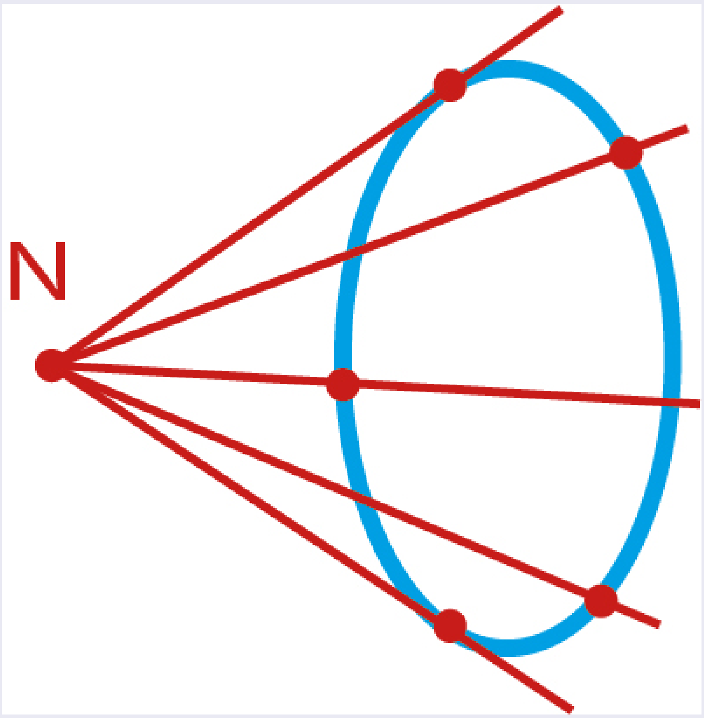
PERFECT

AIM reached:

$$\bar{\ell}(2,3,q) \geq 4 \Longrightarrow \boxed{n < n\frac{s-1}{2}\bar{\ell}(2,3,q) = \mu\bar{\ell}(2,3,q)}$$

Corollary (HYPEROVAL)

q even



$\mathcal{C}$ $[q+2, q-1, 4]_{q}2$ *linear code is* $\left(2, \frac{q+2}{2}\right)$*-MCF*

$\delta_{\mu}(\mathcal{C}, 2) = 1$ ***PERFECT***

Proposition (OVAL)

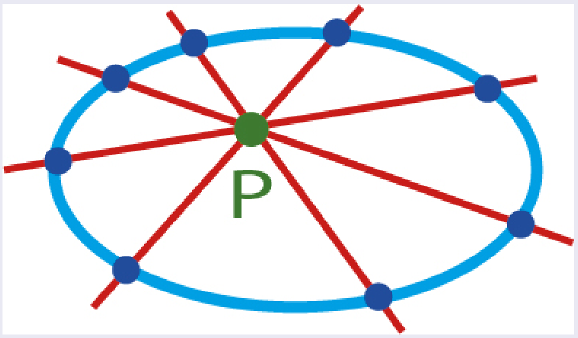
q odd: $\boxed{\text{OVAL in } PG(2,q) \longleftrightarrow \mathcal{C} \quad [q+1, q-2, 4]_q 2}$

$$\Downarrow$$

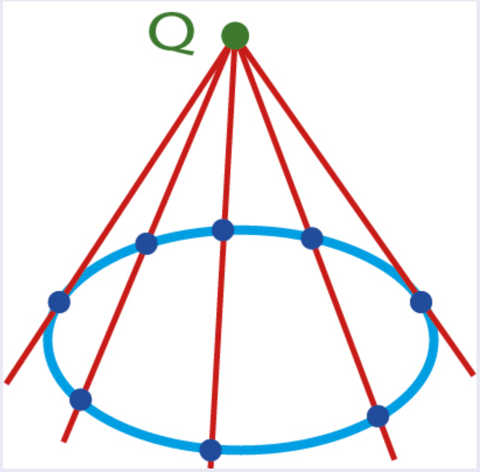
$$\boxed{\begin{array}{c}\mathcal{C} \text{ is a } \left(2, \frac{q-1}{2}\right)\text{-MCF} \\ \delta_\mu(\mathcal{C}, 2) \leq 1 + \frac{1}{q}\end{array}}$$

$q \geq 5 \Longrightarrow$
$q+1 < \mu\bar{\ell}(2,3,q)$

P internal



Q external



$|\text{Bisecants through } P| = \frac{q+1}{2}$

$|\text{Bisecants through } Q| = \frac{q-1}{2} = \mu$

Proposition (HERMITIAN CURVE)

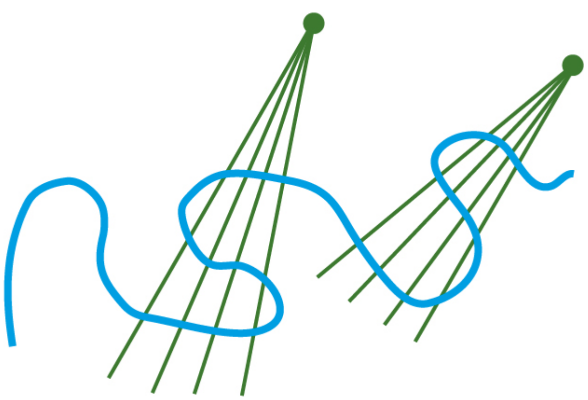
q square

Hermitian Curve in $PG(2,q) \leftrightarrow \mathcal{C} : [\underbrace{q\sqrt{q}+1}_{n}, q\sqrt{q}-2, 3]_q 2$

$$\Downarrow$$

$$\mathcal{C} \text{ is a } \Big(2, \underbrace{\frac{q^2-q}{2}}_{\mu}\Big)\text{-MCF}$$

$$\delta_\mu(\mathcal{C},2) = 1 \ \underline{PERFECT}$$



$\leftarrow (q-\sqrt{q})$ lines

which are $(\sqrt{q}+1)$-secants

$$\Rightarrow (q-\sqrt{q})\binom{\sqrt{q}+1}{2} = \underbrace{\frac{q^2-q}{2}}_{\mu} \text{ bisecants}$$

AIM reached:

$$\bar{\ell}(2,3,q) \geq 4 \Longrightarrow \boxed{n = q\sqrt{q}+1} < 2q^2 - q \leq \boxed{\mu\bar{\ell}(2,3,q)}$$

Proposition (BAER SUBPLANE)

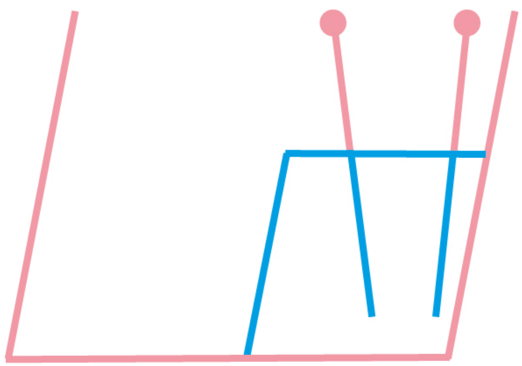
q square

Baer subplane in $PG(2,q) \leftrightarrow \mathcal{C} : [\underbrace{q+\sqrt{q}+1}_{n}, q+\sqrt{q}-2, 3]_q 2$

$\Downarrow$

$\mathcal{C}$ *is a* $\left(2, \underbrace{\frac{q+\sqrt{q}}{2}}_{\mu}\right)$*-MCF*

$\delta_\mu(\mathcal{C}, 2) = 1$ *PERFECT*



unique $(\sqrt{q}+1)$ secant

$\Longrightarrow \binom{\sqrt{q}+1}{2} = \underbrace{\frac{q^2+\sqrt{q}}{2}}_{\mu}$ bisecants

AIM reached:

$\boxed{n = q+\sqrt{q}+1} < 2q + 2\sqrt{q} \le \boxed{\mu\bar{\ell}(2,3,q)}$

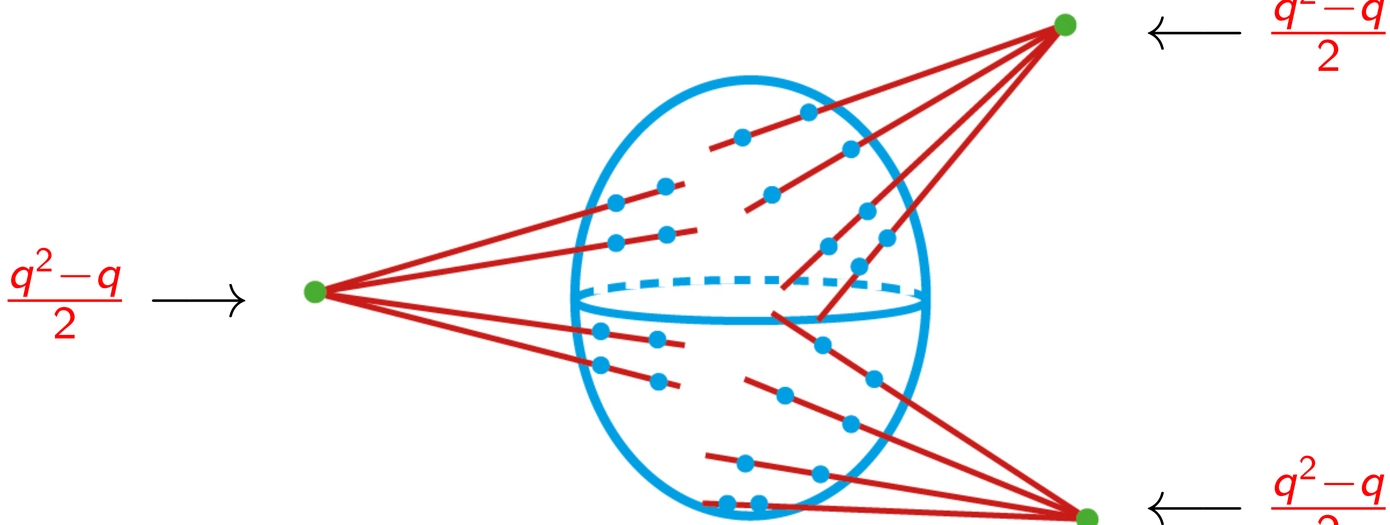
Proposition (ELLIPTIC QUADRIC)

elliptic quadric in $PG(3,q) \longleftrightarrow \mathcal{C} \quad [q^2+1, q^2-3, 4]_q 2$

$$\Downarrow$$

$$\mathcal{C} \text{ is a } \Big(2, \underbrace{\frac{q^2-q}{2}}_{\mu}\Big)\text{-MCF}$$

$\delta_\mu(\mathcal{C},2)=1$ *PERFECT*



WORK IN PROGRESS

- $(1, \mu)$-saturating sets in $PG(2, q)$
  - complete classification for every μ, q small
  - New Constructions $\longrightarrow$ upper bounds for $\ell_\mu(2, 3, q)$

WORK IN PROGRESS

- $(1, \mu)$-saturating sets in $PG(2, q)$
  - complete classification for every μ, q small

    Ex: minimal $(1, \mu)$-saturating sets in $PG(2, 3)$

μ	1	2	3	4	5	6	7	8	9	10	11	12
	$4_1 5_1$	6_4	$6_1 7_1$	$7_1 8_2$	$8_1 9_1$	9_3	$9_1 10_1$	$9_1 10_1$	9_1	11_1	12_1	12_1

  - New Constructions $\longrightarrow$ upper bounds for $\ell_\mu(2, 3, q)$

WORK IN PROGRESS

- $(1,\mu)$-saturating sets in $PG(2,q)$
  - complete classification for every μ, q small
  - New Constructions $\longrightarrow$ upper bounds for $\ell_\mu(2,3,q)$

q	$\ell_2(2,3,q)$	q	$\ell_2(2,3,q)$	q	$\ell_2(2,3,q)$	q	$\ell_2(2,3,q)$	q	$\ell_2(2,3,q)$
97	36	103	38	109	40	127	46	139	50
151	54	157	56	163	58	181	64	193	68
199	70	211	74	223	78	229	80	241	84
269	73	271	94	277	75	281	76	283	98
293	79	307	106	313	84	317	85	331	114
337	90	349	93	353	94	367	126	373	99
379	130	389	103	397	105	401	106	409	108
421	111	433	114	449	118	457	120	461	121
509	133	521	136	541	141	557	145	569	148
577	150	593	154	601	156	613	159	617	160
631	134	641	136	653	169	661	140	673	174
677	175	691	146	701	148	709	183	733	189
751	158	757	195	761	196	761	160	769	198
773	199	797	205	809	208	811	170	821	172
829	213	853	219	857	220	877	225	881	184
911	190	941	196	971	202	991	206	1021	212
1031	214	1051	218	1061	220	1091	226	1151	238
1171	242	1181	244	1201	248	1229	313	1231	254
1237	315	1249	318	1277	325	1279	430	1289	328
1291	434								

WORK IN PROGRESS

- $(1, \mu)$-saturating sets in $PG(2, q)$
  - complete classification for every μ, q small
  - New Constructions $\longrightarrow$ upper bounds for $\ell_\mu(2, 3, q)$
- (ρ, μ)-saturating sets in $PG(N, q)$

General constructions of (ρ, μ)-saturating sets $\nearrow$ translation caps $\searrow$ concatenating constructions

Thank

for your attention!