An Isomorphism between two Arithmetic Fuchsian Groups using Different Edge-Pairings

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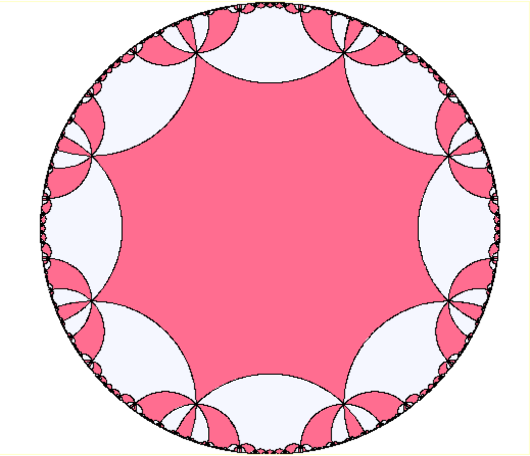
Introduction

- The concept of geometrically uniform codes is strongly dependent on the existence of regular tessellations in homogeneous space.
- The error probability in the hyperbolic plane depends on the genus of a surface and the best performance is achieved with genus ≥ 2 (viewed as edge-pairings of a regular hyperbolic polygon in the hyperbolic plane associated with a hyperbolic tessellations $\{p, q\}$).
- Find Fuchsian groups in order to generate signal constellation.
- For tessellation $\{4g, 4g\}$, with $g \geq 2$, we have shown the arithmetic Fuchsian groups associated with the diametrically opposite edge-pairings and the normal form are isomorphic.

An Example: Tesselation $\{8, 8\}$



Tesselation $\{8, 8\}$ *on the Poincare disc*

Proposal

Obtain arithmetic Fuchsian groups derived from the quaternion orders associated with the tessellation $\{p, q\}$ by using different edge-pairings, and show they are isomorphic.

Motivation

- Construct lattices and geometrically uniform codes using hyperbolic tessellations.
  - E. D. Carvalho, *Identification of lattices from genus of compact surface,* IEEE Trans. Inform. Theory, International Telecommunications Symposium, pp. 146-151, 2006.
  - G. D. Forney, Jr., *Geometrically uniform codes,* IEEE Trans. Inform. Theory, vol. IT 37, pp. 1241-1260, 1991.
- Obtain topological quantum codes algebraically (using Fuchsian groups)
  - C. D. de Albuquerque, R. Palazzo Jr. and E. B. da Silva. *Topological quantum codes on compact surfaces with genus* $g \geq 2$, Journal of Mathematical Physics **50**, pp. 023513, 2009.

Hyperbolic Geometry and Fuchsian Groups

Euclidean models for the hyperbolic plane:

- $\mathbb{H}^2 = \{z \in \mathbb{C} : Im(z) > 0\}$ (upper-half plane)
- $\mathbb{D}^2 = \{z \in \mathbb{C} : |z| < 1\}$ (Poincare disc)

Definition

Let $PSL(2, \mathbb{R})$ be the set of all Möbius transformations, $T : \mathbb{H}^2 \longrightarrow \mathbb{H}^2$, given by $T_A(z) = \frac{az+b}{cz+d}$, where $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$. A Fuchsian group Γ is a discrete subgroup of $PSL(2, \mathbb{R})$.

If $f : \mathbb{H}^2 \longrightarrow \mathbb{D}^2$ given by $f(z) = \frac{zi+1}{z+i}$, then $\Gamma = f^{-1}\Gamma_p f$ is a subgroup of $PSL(2, \mathbb{R})$, where $T_p : \mathbb{D}^2 \longrightarrow \mathbb{D}^2$ and $T_p \in \Gamma_p < PSL(2, \mathbb{C})$ is such that $T_p(z) = \frac{az+b}{\bar{b}z+\bar{a}}$, $a, b \in \mathbb{C}$, $|a|^2 - |b|^2 = 1$. Furthermore, $\Gamma \simeq \Gamma_p$.

Hyperbolic Geometry and Fuchsian Groups

Definition

Let $\mathcal{A} = (\alpha, \beta)_{\mathbb{K}}$ be a quaternion algebra over a field $\mathbb{K}$ with basis $\{1, i, j, k\}$ satisfying $i^2 = \alpha,\ j^2 = \beta$ and $k = ij = -ji$, where $\alpha, \beta \in \mathbb{K}/\{0\}$.

Consider $\varphi : \mathcal{A} \longrightarrow M(2, \mathbb{K}(\sqrt{\alpha}))$ *where*

$$\varphi(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \varphi(i) = \begin{pmatrix} \sqrt{\alpha} & 0 \\ 0 & -\sqrt{\alpha} \end{pmatrix},$$

$$\varphi(j) = \begin{pmatrix} 0 & 1 \\ \beta & 0 \end{pmatrix}, \varphi(k) = \begin{pmatrix} 0 & \sqrt{\alpha} \\ -\beta\sqrt{\alpha} & 0 \end{pmatrix}.$$

So φ *is an isomorphism of* $\mathcal{A} = (\alpha, \beta)_{\mathbb{K}}$ *in the subalgebra* $M(2, \mathbb{K}(\sqrt{\alpha}))$. *Each element of* $\mathcal{A}$ *is identified with*

$$x \longmapsto \varphi(x) = \begin{pmatrix} x_0 + x_1\sqrt{\alpha} & x_2 + x_3\sqrt{\alpha} \\ \beta(x_2 - x_3\sqrt{\alpha}) & x_0 - x_1\sqrt{\alpha} \end{pmatrix}.$$

Hyperbolic Geometry and Fuchsian Groups

Definition

Let $\mathcal{A} = (\alpha, \beta)_{\mathbb{K}}$ be a quaternion algebra and R be a ring of $\mathbb{K}$. An *order* $\mathcal{O}$ in $\mathcal{A}$ is a subring of $\mathcal{A}$ containing 1, equivalently, it is a finitely generated R-module such that $\mathcal{A} = \mathbb{K}\mathcal{O}$.

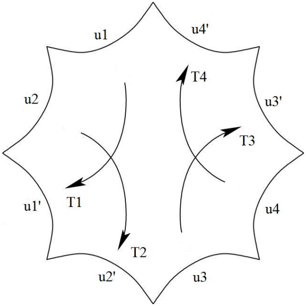
- Considering R a ring of $\mathbb{K}$ and the algebra $\mathcal{A} = (\alpha, \beta)_{\mathbb{K}}$, with $\alpha, \beta \in R$, then

$$\mathcal{O} = \{\alpha_0 + \alpha_1 i + \alpha_2 j + \alpha_3 k : \alpha_0, \alpha_1, \alpha_2, \alpha_3 \in R\},$$

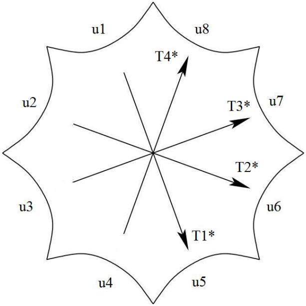
  is an order in $\mathcal{A}$ denoted by $\mathcal{O} = (\alpha, \beta)_R$.
- $\Gamma[\mathcal{A}, \mathcal{O}] \simeq PSL(2, \mathbb{R})$. Therefore, $\Gamma[\mathcal{A}, \mathcal{O}]$ is a Fuchsian group called *arithmetic Fuchsian group*.

Fuchsian Groups Γ_{4g} and Γ^*_{4g}

- Let $\{4g, 4g\}$ be a self-dual tessellation with $g \geq 2$ in the hyperbolic plane and P_{4g} the associated regular hyperbolic polygon.



P_8-normal edge-pairings



P_8-diametrically opposite edge-pairings

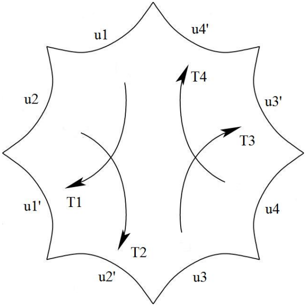
The Fuchsian Group Γ_{4g}

For the normal edge-pairings, we consider the edges of P_{4g} are ordered as follows

$$u_1, u_2, u_1^{'}, u_2^{'}, \ldots, u_{2g-1}, u_{2g}, u_{2g-1}^{'}, u_{2g}^{'},$$

such that

$$T_i(u_i) = u_i^{'}, \ i = 1, \ldots, 2g.$$



The Fuchsian Group Γ_{4g}

If $T_1 \in \Gamma_{4g}$ is such that $T_1(u_1) = u_1^{'}$ then:

$$A_1 = \begin{pmatrix} a & \bar{b} \\ b & \bar{a} \end{pmatrix},$$

with

$$arg(a) = \frac{(g-1)\pi}{2g}, \ |a| = \tan\frac{(2g-1)\pi}{4g}$$

$$\text{and } arg(b) = \frac{-(2g+1)\pi}{4g}, \ |b| = \left(\left(\tan\frac{(2g-1)\pi}{4g}\right)^2 - 1\right)^{\frac{1}{2}}. \quad (1)$$

The Fuchsian Group Γ_{4g}

The remaining generators are obtained by conjugations of the form

$$\begin{cases} A_i = C^{4j+1} A_1 C^{-(4j+1)}, & \text{with } i \text{ even and } j = 0, \ldots, g-1; \\ A_i = C^{4k} A_1 C^{-4k}, & \text{with } i \text{ odd and } k = 1, \ldots, g-1. \end{cases}, \qquad (2)$$

where the $A_i^{'}$s are the transformation matrices associated with the generators $T_i^{'}$s of Γ_{4g}, with $i = 1, \ldots, 2g$ and

$$C = \begin{pmatrix} e^{\frac{i\pi}{4g}} & 0 \\ 0 & e^{-\frac{i\pi}{4g}} \end{pmatrix}$$

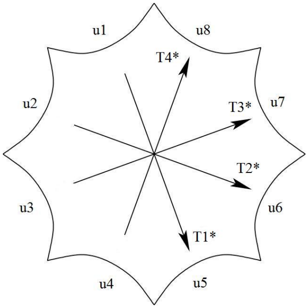
is the matrix corresponding to the elliptic transformation with order $4g$.

$\Gamma_{4g} = \langle T_1, \ldots, T_{2g} \ : \ T_1 \circ T_2 \circ T_1^{-1} \circ T_2^{-1} \circ \cdots \circ T_{2g-1} \circ T_{2g} \circ T_{2g-1}^{-1} \circ T_{2g}^{-1} = Id \rangle.$

The Fuchsian Group Γ_{4g}^*

By using the diametrically opposite edge-pairings, we will consider the Fuchsian group, denoted by Γ_{4g}^*. Let the edges of P_{4g} are ordered as follows

$$u_1, \ldots, u_{4g}, \text{ such that } T_i^*(u_i) = u_{i+2g}, \quad i = 1, \ldots, 2g.$$



The Fuchsian Group Γ_{4g}^*

In the same way, if we have the transformation $T_1^* \in \Gamma_{4g}^*$ and so the corresponding matrix A_1^*, the remaining generators are obtained by conjugations of the form

$$A_i^* = C^{i-1} A_1^* C^{-(i-1)}, \ i = 2, \ldots, 2g. \tag{3}$$

$$\Gamma_{4g}^* = \langle T_1^*, \ldots, T_{2g}^* \ : \ T_1^* \circ (T_2^*)^{-1} \circ \cdots \circ (T_{2g-1}^*)^{-1} \circ T_{2g}^* = Id \rangle.$$

The Fuchsian Group Γ^*_{4g}

Theorem

Let P_p be a hyperbolic regular polygon with p edges and Γ_p the Fuchsian group associated with the tessellation $\{p, q\}$. If $T_1 \in \Gamma_p$ is such that $T_1(u_1) = u_{1+\frac{p}{2}}$ then the matrix A_1 associated with the transformation T_1 is given by

$$A_1 = \begin{pmatrix} \frac{2\cos\frac{\pi}{q}}{2\sin\frac{\pi}{p}} & \frac{\sqrt{2\cos\frac{\pi}{p} + 2\cos\frac{\pi}{q}} \cdot e^{i\left(\frac{p+1}{p}\right)\pi}}{2\sin\frac{\pi}{p}} \\ \frac{\sqrt{2\cos\frac{\pi}{p} + 2\cos\frac{\pi}{q}} \cdot e^{-i\left(\frac{p+1}{p}\right)\pi}}{2\sin\frac{\pi}{p}} & \frac{2\cos\frac{\pi}{q}}{2\sin\frac{\pi}{p}} \end{pmatrix}. \quad (4)$$

Using an appropriate set of edge-pairings of P_p, all the side pairing transformations are obtained by conjugation of the form $T_i = T_{C^{r_i}} \circ T_1 \circ T_{C^{-r_i}}$, where $T_{C^{r_i}}$ is a power of matrix C.

Fuchsian Groups Γ_{4g} and Γ_{4g}^*

- The process of identifying Fuchsian groups derived from a quaternion algebra over a totally real algebraic number field are given by the following results:

Theorem

For each $g = 2^n, 3\cdot 2^n$ and $5 \cdot 2^n$, where $n \in \mathbb{N}$, the elements of a Fuchsian group Γ_{4g} are identified, via isomorphism, with the elements of an order $\mathcal{O} = (\theta, -1)_R$, where $R = \left\{\frac{\delta}{2^m} : \delta \in \mathbb{Z}[\theta] \text{ e } m \in \mathbb{N}\right\}$ and

$$\theta = \begin{cases} \sqrt{2+\sqrt{2+\ldots+\sqrt{2}}} & \text{for } g = 2^n; \\ \sqrt{2+\sqrt{2+\ldots+\sqrt{2+\sqrt{3}}}} & \text{for } g = 3\cdot 2^n; \\ \sqrt{2+\sqrt{2+\ldots+\sqrt{2+\frac{\sqrt{10+2\sqrt{5}}}{2}}}} & \text{for } g = 5\cdot 2^n. \end{cases} \tag{5}$$

Fuchsian Groups Γ_{4g} and Γ_{4g}^*

Theorem

For each $g = 2^n, 3 \cdot 2^n$ and $5 \cdot 2^n$, with $n \in \mathbb{N}$, the Fuchsian group Γ_{4g}, associated with the hyperbolic polygon P_{4g}, is derived from a quaternion algebra $\mathcal{A} = (\theta, -1)_{\mathbb{K}}$, over the number field $\mathbb{K} = \mathbb{Q}(\theta)$, where $[\mathbb{K} : \mathbb{Q}] = 2^n, 2^{n+1}$ and 2^{n+2}, respectively, and θ is as in (5).

Reminding the proposal:

The purpose is to show that there is an isomorphism between arithmetic Fuchsian groups derived from quaternion algebra each fixed tessellation $\{p, q\}$.

Following, we prove this for the groups Γ_{4g} and Γ_{4g}^{*}.

Isomorphism between Arithmetic Fuchsian Groups

Lemma

Let $\Gamma \subset PSL(2,\mathbb{R})$ be a finitely generated Fuchsian group with generators $G_1, \ldots G_l$. Then

$$G_i = \frac{1}{2^s} \begin{pmatrix} x_i + y_i\sqrt{\theta} & z_i + w_i\sqrt{\theta} \\ -z_i + w_i\sqrt{\theta} & x_i - y_i\sqrt{\theta} \end{pmatrix}, \qquad (6)$$

where $G_i \in M(2, \mathbb{K}(\sqrt{\theta}))$, $s \in \mathbb{N}$, $\theta, x_i, y_i, z_i, w_i \in \mathbb{K}$, with $i = 1, \ldots, l$ and $\mathbb{K}$ a totally real algebraic number field. Furthermore, any element $T \in \Gamma$ has the same form as that of the generators of Γ.

Isomorphism between Arithmetic Fuchsian Groups

Theorem

Let $\Gamma_{4g}, \Gamma_{4g}^* \subset PSL(2, \mathbb{C})$ *be arithmetic Fuchsian groups associated with the tessellation* $\{4g, 4g\}$ *using the normal edge-paring and the diametrically opposite edge-pairings, respectively. Then* $\Gamma_{4g} \simeq \Gamma_{4g}^*$.

Isomorphism between Arithmetic Fuchsian Groups

Proof.

- Fixed a genus g and consider Γ_1 and Γ_2 as Fuchsian groups in $\mathbb{H}^2$.
- So, $\Gamma_1 = f^{-1}\Gamma_{4g}f$ and $\Gamma_2 = f^{-1}\Gamma_{4g}^{*}f$, where $f : z \longmapsto \frac{zi+1}{z+i}$.
- Consider the isomorphism of groups $\phi_1 : \Gamma_1 \longrightarrow \Gamma_{4g}$ and $\phi_2 : \Gamma_2 \longrightarrow \Gamma_{4g}^{*}$ given by $\phi_1(T) = \phi_2(T) = f^{-1}Tf$.
- So, $\Gamma_1 \simeq \Gamma_{4g}$ and $\Gamma_2 \simeq \Gamma_{4g}^{*}$.
- For $i = 1, \ldots, 2g$ we have that $G_i = f^{-1}A_i f$ and $G_i^{*} = f^{-1}A_i^{*}f$.
- Since $\Gamma_1, \Gamma_2 \subset PSL(2, \mathbb{R})$, by Lemma it follows that both generators G_i and G_i^{*} are of the form given in (6) and $G_i, G_i^{*} \subset M(2, \mathbb{Q}(\sqrt{\theta}))$.
- There exists an isomorphism $\psi : \Gamma_1 \longrightarrow \Gamma_2$ in which $\psi(G_i) = G_i^{*}$.
- By the chain of isomorphisms $\Gamma_{4g} \simeq \Gamma_1 \simeq \Gamma_2 \simeq \Gamma_{4g}^{*}$.

Examples

Example

Let P_8 be the regular hyperbolic polygon associated with the tessellation $\{8, 8\}$. Let us consider the normal form for the edge-pairings of P_8. Using the equalities in (1) and (2), and the fact that $G_i = f^{-1} A_i f$, with $i = 1, \cdots, 4$ we obtain the following generators of the arithmetic Fuchsian group Γ_8:

Examples

$$G_1 = \begin{pmatrix} \frac{x_1 - x_1\sqrt[4]{2}}{2} & \frac{x_1 - y_1\sqrt[4]{2}}{2} \\ \frac{-x_1 - y_1\sqrt[4]{2}}{2} & \frac{x_1 + x_1\sqrt[4]{2}}{2} \end{pmatrix} = \varphi\left(\frac{x_1}{2} - \frac{x_1}{2}i + \frac{x_1}{2}j - \frac{y_1}{2}k\right), \quad G_2 = \begin{pmatrix} \frac{x_1 + x_1\sqrt[4]{2}}{2} & \frac{x_1 + y_1\sqrt[4]{2}}{2} \\ \frac{-x_1 + y_1\sqrt[4]{2}}{2} & \frac{x_1 - x_1\sqrt[4]{2}}{2} \end{pmatrix} = \varphi\left(\frac{x_1}{2} + \frac{x_1}{2}i + \frac{x_1}{2}j + \frac{y_1}{2}k\right),$$

$$G_3 = \begin{pmatrix} \frac{x_1 - x_1\sqrt[4]{2}}{2} & \frac{-x_1 + y_1\sqrt[4]{2}}{2} \\ \frac{x_1 + y_1\sqrt[4]{2}}{2} & \frac{x_1 + x_1\sqrt[4]{2}}{2} \end{pmatrix} = \varphi\left(\frac{x_1}{2} - \frac{x_1}{2}i - \frac{x_1}{2}j + \frac{y_1}{2}k\right), \quad G_4 = \begin{pmatrix} \frac{x_1 + x_1\sqrt[4]{2}}{2} & \frac{-x_1 - y_1\sqrt[4]{2}}{2} \\ \frac{x_1 - y_1\sqrt[4]{2}}{2} & \frac{x_1 - x_1\sqrt[4]{2}}{2} \end{pmatrix} = \varphi\left(\frac{x_1}{2} + \frac{x_1}{2}i - \frac{x_1}{2}j - \frac{y_1}{2}k\right),$$

where $x_1 = 2 + \sqrt{2}$ and $y_1 = \sqrt{2}$.

Examples

Hence, the quaternion order associated with the Fuchsian group Γ_8 is $\mathcal{O} = (\sqrt{2}, -1)_R$, where $R = \{\frac{\delta}{2^m} : \delta \in \mathbb{Z}[\sqrt{2}] \text{ and } m \in \mathbb{N}\}$, and Γ_8 is derived from the quaternion algebra $\mathcal{A} = (\sqrt{2}, -1)_{\mathbb{K}}$, with $\mathbb{K} = \mathbb{Q}(\sqrt{2})$ and $[\mathbb{K} : \mathbb{Q}] = 2$.

Example

Let P_8 be the regular hyperbolic polygon associated with the tessellation $\{8, 8\}$. Let us consider the diametrically opposite edge-pairings of P_8. Using the equalities in (3) and (4), and the fact that $G_i^* = f^{-1} A_i f,\ i = 1, \cdots, 4$ we obtain the following generators of the arithmetic Fuchsian group Γ_8^*:

Examples

$$G_1^* = \begin{pmatrix} \frac{x_1+y_1\sqrt[4]{2}}{2} & \frac{-w_1\sqrt[4]{2}}{2} \\ \frac{-w_1\sqrt[4]{2}}{2} & \frac{x_1-y_1\sqrt[4]{2}}{2} \end{pmatrix} = \varphi\left(\frac{x_1}{2} + \frac{y_1}{2}i - \frac{w_1}{2}k\right), \quad G_2^* = \begin{pmatrix} \frac{x_1-w_1\sqrt[4]{2}}{2} & \frac{y_1\sqrt[4]{2}}{2} \\ \frac{y_1\sqrt[4]{2}}{2} & \frac{x_1+w_1\sqrt[4]{2}}{2} \end{pmatrix} = \varphi\left(\frac{x_1}{2} - \frac{w_1}{2}i + \frac{y_1}{2}k\right),$$

$$G_3^* = \begin{pmatrix} \frac{x_1-w_1\sqrt[4]{2}}{2} & \frac{-y_1\sqrt[4]{2}}{2} \\ \frac{-y_1\sqrt[4]{2}}{2} & \frac{x_1+w_1\sqrt[4]{2}}{2} \end{pmatrix} = \varphi\left(\frac{x_1}{2} - \frac{w_1}{2}i - \frac{y_1}{2}k\right), \quad G_4^* = \begin{pmatrix} \frac{x_1-y_1\sqrt[4]{2}}{2} & \frac{-w_1\sqrt[4]{2}}{2} \\ \frac{-w_1\sqrt[4]{2}}{2} & \frac{x_1+y_1\sqrt[4]{2}}{2} \end{pmatrix} = \varphi\left(\frac{x_1}{2} - \frac{y_1}{2}i - \frac{w_1}{2}k\right),$$

where $x_1 = 2 + 2\sqrt{2}, y_1 = \sqrt{2}$ and $w_1 = 2 + \sqrt{2}$.

Examples

Hence, the quaternion order associated with the Fuchsian group Γ_8^* derived from the quaternion algebra $\mathcal{A} = (\sqrt{2}, -1)_{\mathbb{K}}$, is $\mathcal{O} = (\sqrt{2}, -1)_R$, where $R = \{\frac{\delta}{2^m} : \delta \in \mathbb{Z}[\sqrt{2}]$ and $m \in \mathbb{N}\}$, $\mathbb{K} = \mathbb{Q}(\sqrt{2})$ and $[\mathbb{K} : \mathbb{Q}] = 2$.

Observation: *These results extend to other tessellations other than the self-dual tessellation* $\{4g, 4g\}$.

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THAND YOU!

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