Bounds on List Decoding Gabidulin Codes

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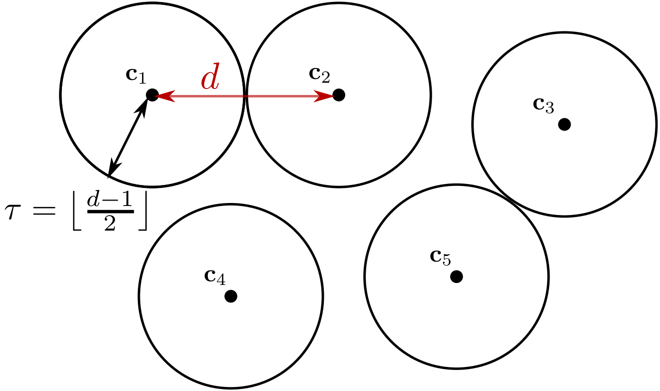
Unique Decoding

For a code $\mathcal{C}$ of length n, dimension k and minimum distance d, unique decoding is possible up to $\tau = \left\lfloor \frac{d-1}{2} \right\rfloor$.



What about decoding algorithms for Gabidulin codes?
Similar to Reed–Solomon codes?

Reed–Solomon vs. Gabidulin Codes — Algorithms

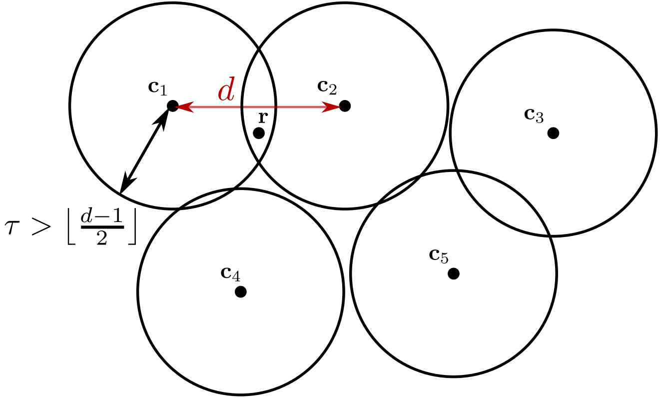
Decoding up to half the minimum distance $\tau = \left\lfloor \frac{d-1}{2} \right\rfloor$

	Reed–Solomon Codes	Gabidulin Codes
System of equations	Peterson, ...	Gabidulin
Shift–Register Synthesis	Berlekamp–Massey	Paramonov–Tretjakov, Richter–Plass
Euclidean Algorithm	Sugiyama, ...	Gabidulin
Interpolation	Welch–Berlekamp	Loidreau
⋮	⋮	⋮

Many parallels between Reed–Solomon and Gabidulin codes!

List Decoding

For a code $\mathcal{C}$ of length n, dimension k and minimum distance d, there can be several codewords in a ball of radius $\tau > \left\lfloor \frac{d-1}{2} \right\rfloor$.



What about decoding algorithms for Gabidulin codes?

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Reed–Solomon vs. Gabidulin Codes — Algorithms

Decoding beyond half the minimum distance $\tau > \left\lfloor \frac{d-1}{2} \right\rfloor$

	Reed–Solomon Codes	Gabidulin Codes
Interpolation (List Decoding)	Sudan Guruswami–Sudan (and many accelerations)	?
Syndrome-based (Unique Decoding)	Schmidt–Sidorenko	

Is polynomial–time list decoding possible for Gabidulin codes?

Outline

1. Rank Metric Codes
2. Problem Statement and Overview of Bounds
3. New Bounds on the List Size
   - Lower Bound
   - Upper Bound
4. Conclusion

Outline

Rank Metric

- Let $\mathcal{B}$ be a basis of $\mathbb{F}_{q^m}$ over $\mathbb{F}_q$ where q is a power of a prime
- Each vector $\mathbf{x} \in \mathbb{F}_{q^m}^n$ can be mapped on a matrix $\mathbf{X} \in \mathbb{F}_q^{m \times n}$
- Rank norm: $\mathrm{rank}_q(\mathbf{x})$ = rank of $\mathbf{X}$ over $\mathbb{F}_q$

Minimum Rank Distance of a block code $\mathcal{C}$:

- $d = \min\{\mathrm{rank}_q(\mathbf{c}) \mid \mathbf{c} \in \mathcal{C}, \mathbf{c} \neq \mathbf{0}\} \leq n - k + 1$
- Codes with $d = n - k + 1$ are called Maximum Rank Distance (MRD) codes

Linearized Polynomial over $\mathbb{F}_{q^m}$

- $f(x) \stackrel{\text{def}}{=} \sum_{i=0}^{d_f} f_i x^{[i]} = \sum_{i=0}^{d_f} f_i x^{q^i}$ with $f_i \in \mathbb{F}_{q^m}$.
- If $f_{d_f} \neq 0$, define the q-degree: $\deg_q f(x) = d_f$.

Gabidulin Codes

Introduced by *Delsarte* (1978), *Gabidulin* (1985), *Roth* (1991)

- A linear Gabidulin code $\mathcal{G}(n,k)$ of length $n \leq m$ and dimension k over $\mathbb{F}_{q^m}$ is defined by

  $$\mathcal{G}(n,k) \stackrel{\text{def}}{=} \{\mathbf{c} = (f(\alpha_0), f(\alpha_1), \dots, f(\alpha_{n-1}) \big| \deg_q f(x) < k)\},$$

  where the fixed elements $\alpha_0, \dots, \alpha_{n-1} \in \mathbb{F}_{q^m}$ are linearly independent over $\mathbb{F}_q$.

Minimum Rank Distance of a Gabidulin Code

- $d = \min\{\text{rank}_q(\mathbf{c}) \mid \mathbf{c} \in \mathcal{G}, \mathbf{c} \neq \mathbf{0}\} = n - k + 1.$

Outline

Problem Statement

Is polynomial–time list decoding possible for Gabidulin codes?

Problem (Maximum List Size)

Let the Gabidulin code $\mathcal{G}(n,k)$ over $\mathbb{F}_{q^m}$ with $n \leq m$ and $d = n-k+1$ be given. Let $\tau < d$. Find a lower and upper bound on the maximum number of codewords ℓ in the ball of rank radius τ around $\mathbf{r} = (r_0\ r_1\ \dots\ r_{n-1}) \in \mathbb{F}_{q^m}^n$. Hence, find a bound on

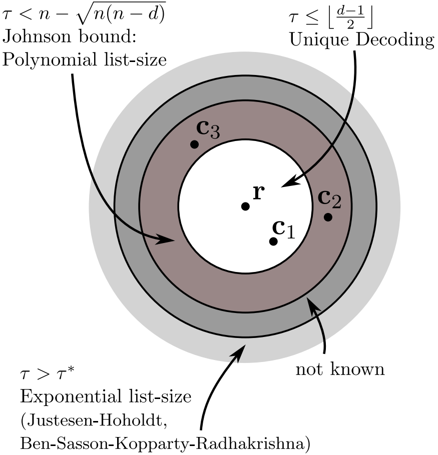
$$\ell \stackrel{\text{def}}{=} \max_{\mathbf{r} \in \mathbb{F}_{q^m}^n} \left(|\mathcal{B}_\tau(\mathbf{r}) \cap \mathcal{G}(n,k)| \right).$$

Interpretation:

- Lower exponential bound: no polynomial–time list decoding,
- Upper polynomial bound: polynomial–time list decoding might exist.

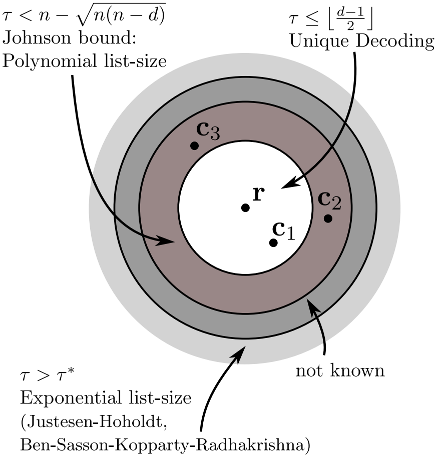
Bounds on the Maximal List-Size

Reed–Solomon codes

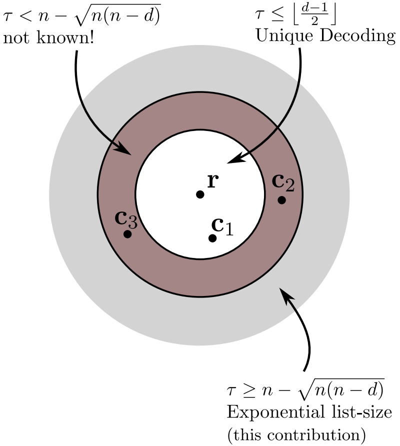


Bounds on the Maximal List-Size

Reed–Solomon codes



Gabidulin codes



Outline

A Lower Bound on the List Size

Theorem (Lower Bound on the List Size)

Let the Gabidulin code $\mathcal{G}(n,k)$ over $\mathbb{F}_{q^m}$ with $n \leq m$ and $d = n-k+1$ be given. Let $\tau < d$. Then, there exists a word $\mathbf{r} \in \mathbb{F}_{q^m}^n$ such that

$$\ell \geq |\mathcal{B}_\tau(\mathbf{r}) \cap \mathcal{G}(n,k)| \geq \frac{\begin{bmatrix} n \\ n-\tau \end{bmatrix}}{(q^m)^{n-\tau-k}} \geq q^m q^{\tau(m+n)-\tau^2-md},$$

and for the special case of $n = m$: $\ell \geq q^n q^{2n\tau-\tau^2-nd}$.

- For $n = m$ this is $\ell \geq q^{n(1-\epsilon)} \cdot q^{2n\tau-\tau^2-nd+n\epsilon}$
- Exponential in n if $\tau \geq n - \sqrt{n(n-d+\epsilon)}$ and $0 \leq \epsilon < 1$.

A Lower Bound on the List Size

Theorem (Lower Bound on the List Size)

Let the Gabidulin code $\mathcal{G}(n,k)$ *over* $\mathbb{F}_{q^m}$ *with* $n \leq m$ *and* $d = n-k+1$ *be given. Let* $\tau < d$*. Then, there exists a word* $\mathbf{r} \in \mathbb{F}_{q^m}^n$ *such that*

$$\ell \geq |\mathcal{B}_\tau(\mathbf{r}) \cap \mathcal{G}(n,k)| \geq \frac{\begin{bmatrix} n \\ n-\tau \end{bmatrix}}{(q^m)^{n-\tau-k}} \geq q^m q^{\tau(m+n)-\tau^2-md},$$

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A Lower Bound on the List Size – Proof

Proof (i)

- $\mathcal{P}^* =$ all monic linearized polynomials with $\deg_q = n - \tau$ and a root space over $\mathbb{F}_{q^n}$ of dimension $n - \tau > k - 1$
- $|\mathcal{P}^*| = \begin{bmatrix} n \\ n-\tau \end{bmatrix}$
- $\mathcal{P} =$ subset of $\mathcal{P}^*$ such that all q-monomials of q-degree greater than or equal to k have the same coefficients
- there are $(q^m)^{n-\tau-k}$ possibilities to choose the highest $n - \tau - (k-1)$ coefficients
- there exist coefficients such that $|\mathcal{P}| \geq \frac{\begin{bmatrix} n \\ n-\tau \end{bmatrix}}{(q^m)^{n-\tau-k}}$
- For any $f(x), g(x) \in \mathcal{P}$, $\deg_q(f(x) - g(x)) < k$, is evaluation polynomial of a codeword of $\mathcal{G}(n, k)$

Proof (ii)

- Let $f(x), g(x) \in \mathcal{P}$
- Let $\mathcal{A} = \{\alpha_0, \alpha_1, \dots, \alpha_{n-1}\}$ be a basis of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$
- Let $\mathbf{r} = (r_0 \; r_1 \; \dots \; r_{n-1}) = (f(\alpha_0) \; f(\alpha_1) \; \dots \; f(\alpha_{n-1}))$
- Let $\mathbf{c}$ be the evaluation of $f(x) - g(x)$ at $\mathcal{A}$
- Then, $\mathbf{r} - \mathbf{c}$ is the evaluation of $f(x) - f(x) + g(x) = g(x) \in \mathcal{P}$, whose root space has dimension $n - \tau$ and all roots are in $\mathbb{F}_{q^n}$
- $\dim \ker(\mathbf{r} - \mathbf{c}) = n - \tau$ and $\dim \operatorname{im}(\mathbf{r} - \mathbf{c}) = \operatorname{rk}(\mathbf{r} - \mathbf{c}) = \tau$

Therefore, for *any* $g(x) \in \mathcal{P}$, the evaluation of $f(x) - g(x)$ is a codeword from $\mathcal{G}(n, k)$ and has rank distance τ from $\mathbf{r}$.

$$\Longrightarrow \ell \geq |\mathcal{P}| \geq \frac{\begin{bmatrix} n \\ n-\tau \end{bmatrix}}{(q^m)^{n-\tau-k}}. \qquad \square$$

An Upper Bound on the List Size

Theorem (Upper Bound on the List Size)

Let the Gabidulin code $\mathcal{G}(n,k)$ *over* $\mathbb{F}_{q^m}$ *with* $n \leq m$ *and* $d = n-k+1$ *be given. Let* $\tau < d$*. Then, for any word* $\mathbf{r} \in \mathbb{F}_{q^m}^n$ *and hence, for the maximum list size, the following holds*

$$\ell = \max_{\mathbf{r}\in\mathbb{F}_{q^m}^n} \left(|\mathcal{B}_\tau(\mathbf{r}) \cap \mathcal{G}|\right) \leq \sum_{t=\left\lfloor\frac{d-1}{2}\right\rfloor+1}^{\tau} \frac{\begin{bmatrix} n \\ 2t+1-d \end{bmatrix}}{\begin{bmatrix} t \\ 2t+1-d \end{bmatrix}}$$

$$\leq 4 \sum_{t=\left\lfloor\frac{d-1}{2}\right\rfloor+1}^{\tau} q^{(2t-d+1)(n-t)}$$

- Exponential in $n \leq m$ for any $\tau > \lfloor (d-1)/2 \rfloor$
- Does not provide any conclusion if polynomial-time list decoding is possible or not up to the Johnson bound.

Outline

Conclusion

We have provided two bounds on the list size of Gabidulin codes.

The upper bound

- is exponential in n,
- uses subspace properties.

The lower bound

- is based on the evaluation of linearized polynomials,
- shows that polynomial-time list decoding is not possible for $\tau \geq n - \sqrt{n(n-d+\epsilon)}$.