Classification of the odd sets in PG$(4, 4)$

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Contents

1. Geometric approach

$\mathcal{C}$: $[n, k, d]_4$ code, $k \geq 3$

$G = [g_1^\top, \cdots, g_k^\top]^\top$: a generator matrix of $\mathcal{C}$

$\Sigma :=$PG$(k-1, 4)$: the projective space of dimension $k-1$ over $\mathbb{F}_4$

For $P = \mathbf{P}(p_1, \ldots, p_k) \in \Sigma$ we define the weight of P with respect to G, denoted by $w_G(P)$, as

$$w_G(P) = wt(p_1 g_1 + \cdots + p_k g_k).$$

A hyperplane H of Σ is defined by a non-zero vector $h = (h_0, \ldots, h_{k-1}) \in \mathbb{F}_4^k$ as

$$H = \{P = \mathbf{P}(p_0, \ldots, p_{k-1}) \in \Sigma \mid h_0 p_0 + \cdots + h_{k-1} p_{k-1} = 0\}.$$

h is called a defining vector of H.

Let $F_d = \{P \in \Sigma \mid w_G(P) = d\}$.

Lem 1. (Maruta 2008)

$\mathcal{C}$ is extendable $\Leftrightarrow$ there exists a hyperplane H of Σ s.t. $F_d \cap H = \emptyset$.

Moreover, $[G, h]$ generates an extension of $\mathcal{C}$, where $h^{\mathsf{T}} \in \mathbb{F}_q^k$ is a defining vector of H.

Now, let

$$F_0 = \{P \in \Sigma \mid w_G(P) \equiv 0 \pmod 4\},$$
$$F_1 = \{P \in \Sigma \mid w_G(P) \not\equiv 0, d \pmod 4\},$$

$(\Phi_0, \Phi_1) = (|F_0|, |F_1|)$: diversity of $\mathcal{C}$

Assume $|F_1| = 0$.

Let $\mathscr{K} = F_0$.

Π_t : a t-flat in Σ.

For $\mathscr{K} \subset \Sigma$,

hyperplane π: i-hyperplane if $|\pi \cap \mathscr{K}| = i$

solid Δ: i-solid if $|\Delta \cap \mathscr{K}| = i$.

plane δ : i-plane if $|\delta \cap \mathscr{K}| = i$.

line l : i-line if $|l \cap \mathscr{K}| = i$.

point P : 1-point if $P \in \mathscr{K}$.

0-point if $P \notin \mathscr{K}$.

Lem 2.

$\mathcal{C}$ is extendable if there exists a hyperplane H of Σ such that $H \subset F_0$.

The following two theorems can be proved applying this lemma.

Thm 3. (Yoshida & Maruta 2009)
Let $\mathcal{C}$ be an $[n, k, d]_4$ code with diversity $(\Phi_0, 0)$, $k \geq 3$, $d \equiv 2 \pmod 4$. Then $\mathcal{C}$ is extendable if $\Phi_0 = \theta_{k-2}$ or $(\theta_{k-1} + \theta_{k-2} + 4^{k-2})/2$ where $\theta_j = (4^{j+1} - 1)/(4 - 1)$.

Thm 4.

Let $\mathcal{C}$ be an $[n, k, d]_4$ code with diversity $(\Phi_0, 0)$, $k \geq 3$, $d \equiv 2 \pmod 4$. Then $\mathcal{C}$ is extendable if $\Phi_0 = (\theta_{k-1} + \theta_{k-2})/2$, where $\theta_j = (4^{j+1} - 1)/3$.

This can be proved by using our results.

Lem 5. (Maruta 2008)

For a line $L = \{P_0, P_1, \cdots, P_5\}$ in Σ,

it hold that

$$\sum_{i=0}^{5} w_G(P_i) \equiv 0 \pmod 4.$$

Since $w_G(P_i) \equiv 0$ or $2 \pmod 4$,

$|L \setminus \mathscr{K}| \in \{0, 2, 4\}$, i.e. $|L \cap \mathscr{K}| \in \{1, 3, 5\}$.

So, $\mathscr{K}$ has only 1-*lines*, 3-*lines*, 5-*lines*.

2. Odd sets in $PG(d,4)$

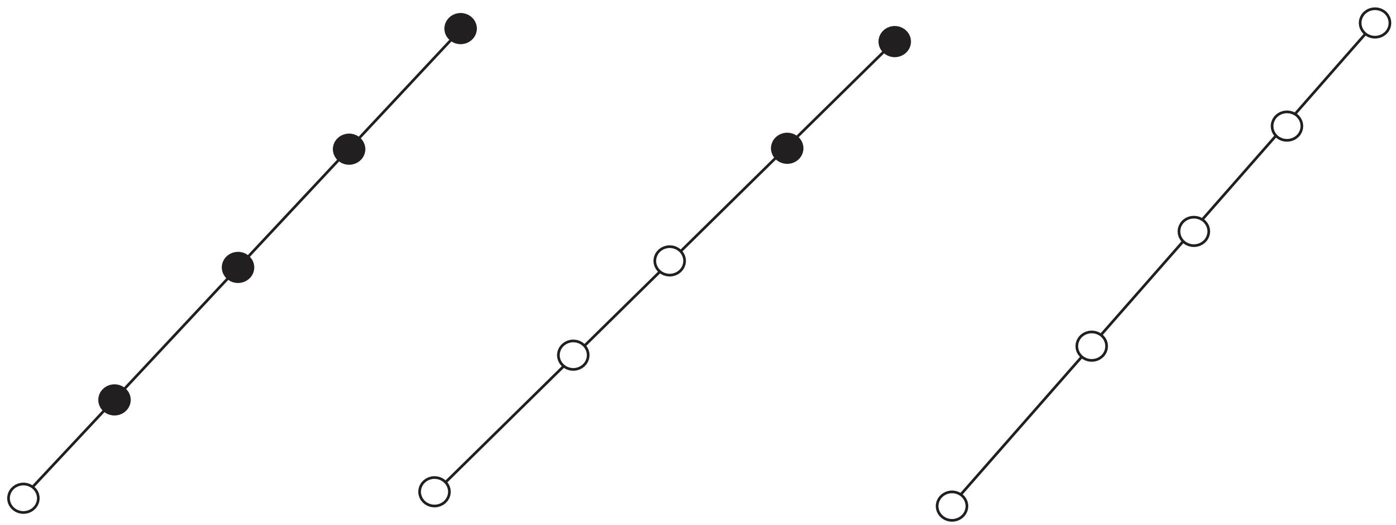
For $\mathscr{K} \subset PG(d,4)$,

$\mathscr{K}$ is an odd set,

if $\mathscr{K}$ has only 1-*lines*, 3-*lines*, 5-*lines*.

O_d is the set of odd sets in $PG(d,4)$.

1-*line*, 3-*line*, 5-*line*



$\circ \in \mathscr{K}$, $\bullet \notin \mathscr{K}$.

Known results on odd sets in $PG(d, 4)$

- Hirschfeld and Hubaut (1980) characterized all odd sets in $PG(3, 4)$.
- Sherman (1983) gave an algebraic characterization of odd sets in $PG(d, 4)$.

This research...

1. We classify odd sets in $PG(4,4)$ by way of Sherman's method.
2. We prove Thm 4 applying our results on odd sets.

Spectrum

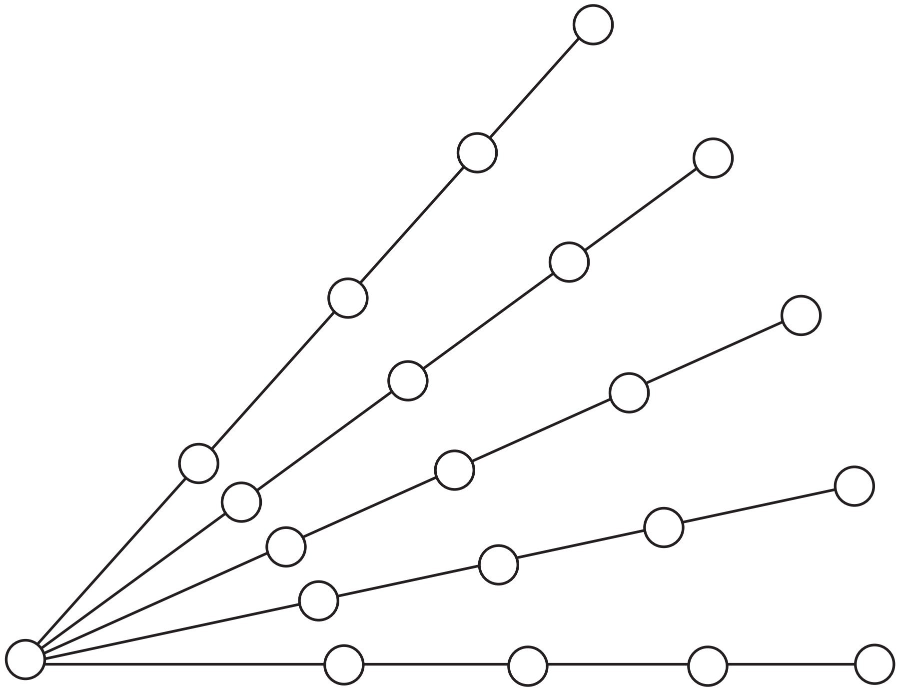
c_i : the number of i-hyperplanes.

The list of c_i's is *spectrum* of $\mathscr{K}$.

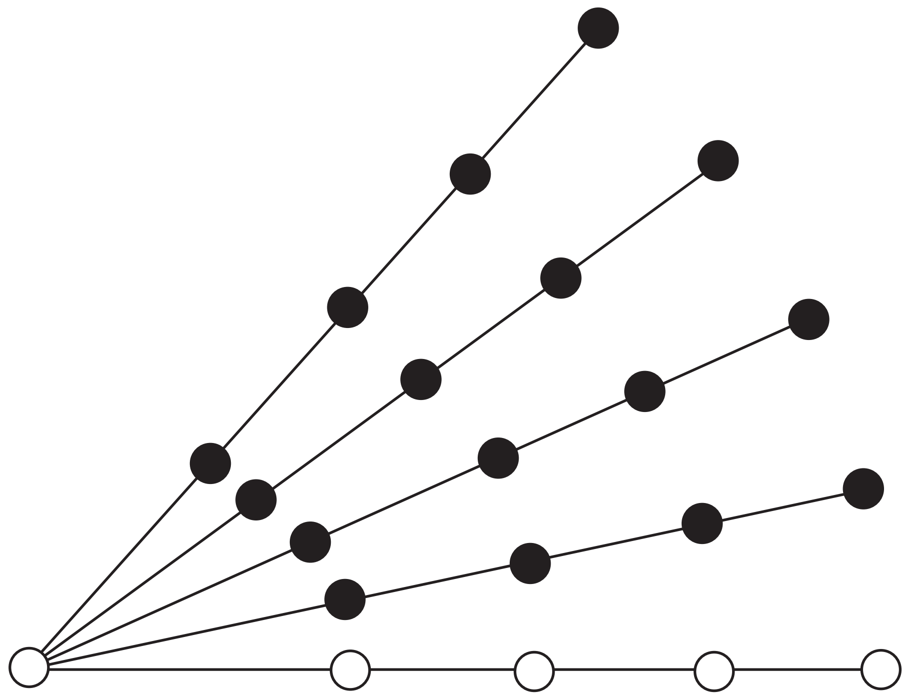
Odd sets in $PG(2,4)$ [1]

Type	Φ_0	$c_1^{(2)}$	$c_3^{(2)}$	$c_5^{(2)}$
Π_2	21			21
Π_1	5	20		1
$\Pi_0\mathscr{U}_1$	13	2	16	3
$\mathscr{U}_2$	9	9	12	
$\mathscr{F}_a$	7	14	7	
$\mathscr{P}$	11	5	15	1
$\mathscr{O}^c$	15		15	6

$$\Pi_2 \;:\; (\Phi_0; c_1, c_3, c_5) = (21; 0, 0, 21)$$

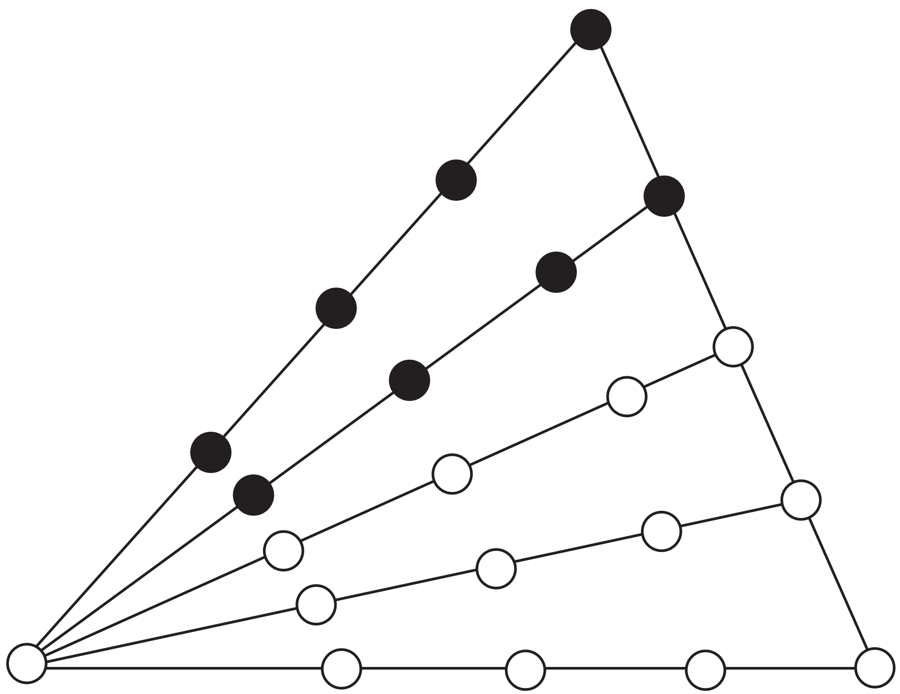


$\Pi_1 \quad : (\Phi_0; c_1, c_3, c_5) = (5; 20, 0, 1)$



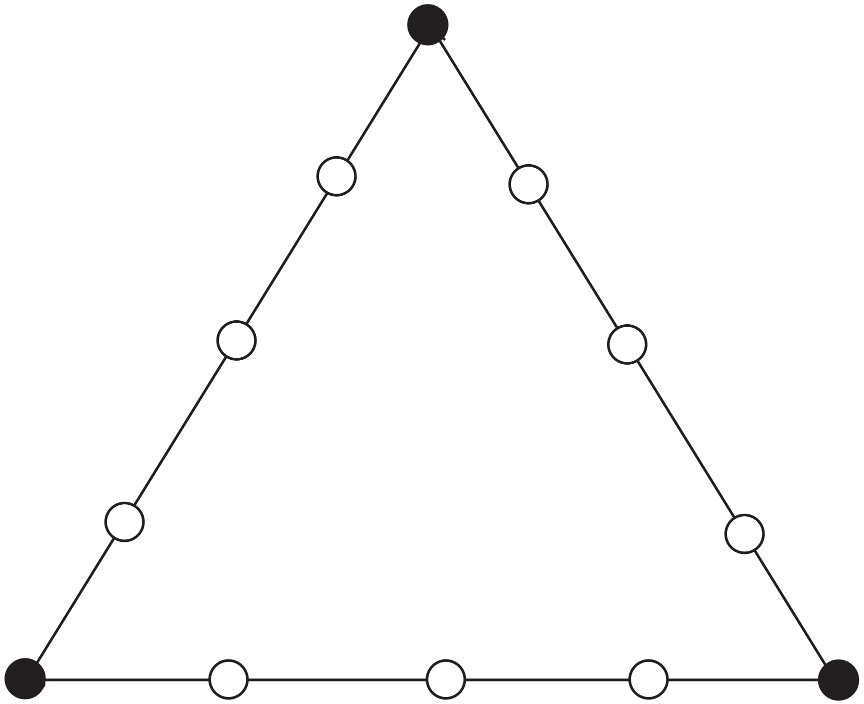
$\mathbf{V}(x_0^3).$

$\Pi_0\mathscr{U}_1 \quad : (\Phi_0; c_1, c_3, c_5) = (13; 2, 16, 3)$



$\mathbf{V}(x_0^3 + x_1^3).$

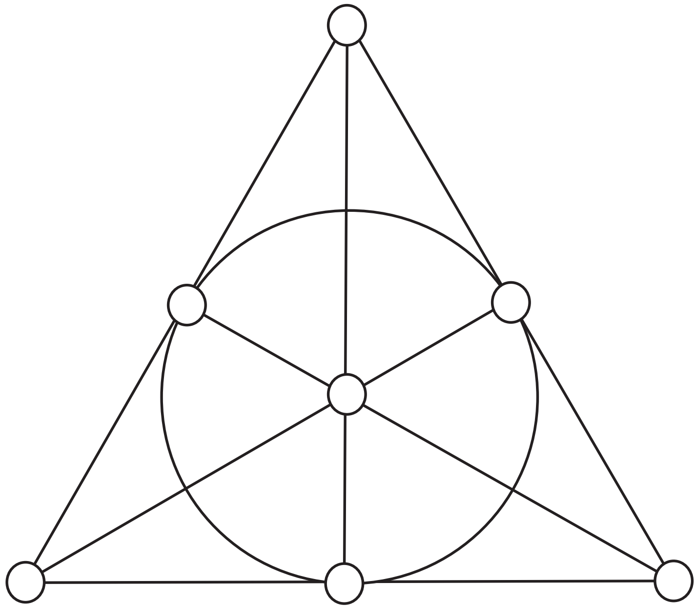
$\mathscr{U}_2 \quad : (\Phi_0; c_1, c_3, c_5) = (9; 9, 12, 0)$



$\mathbf{V}(x_0^3 + x_1^3 + x_2^3)$.

Hermitian arc

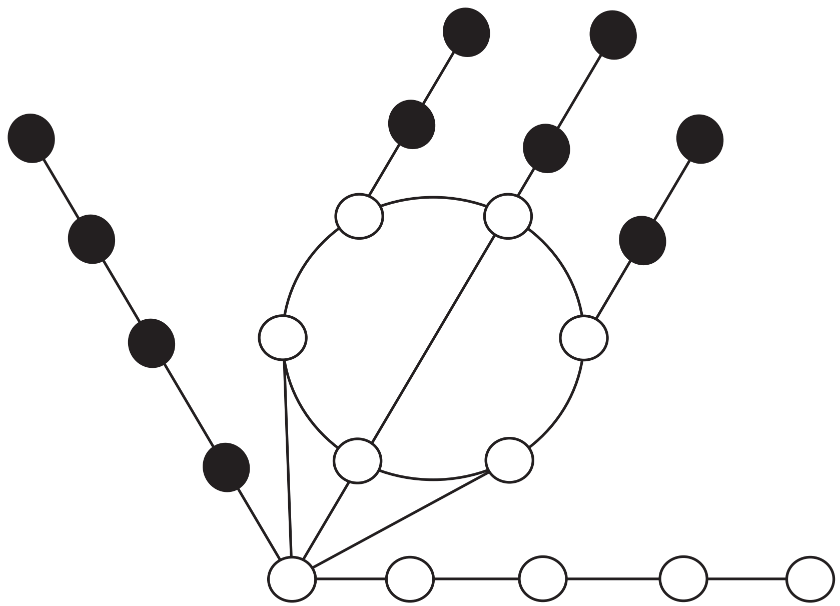
$\mathscr{F}_a \quad : (\Phi_0; c_1, c_3, c_5) = (7; 14, 7, 0)$

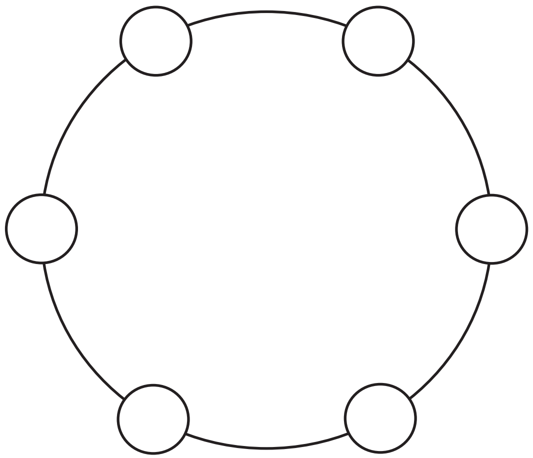


$$\mathbf{V}(x_0^2 x_1^2 x_2^2 + x_0 x_1 x_2 + x_0^3 + x_1^3 + x_2^3).$$

Fano plane

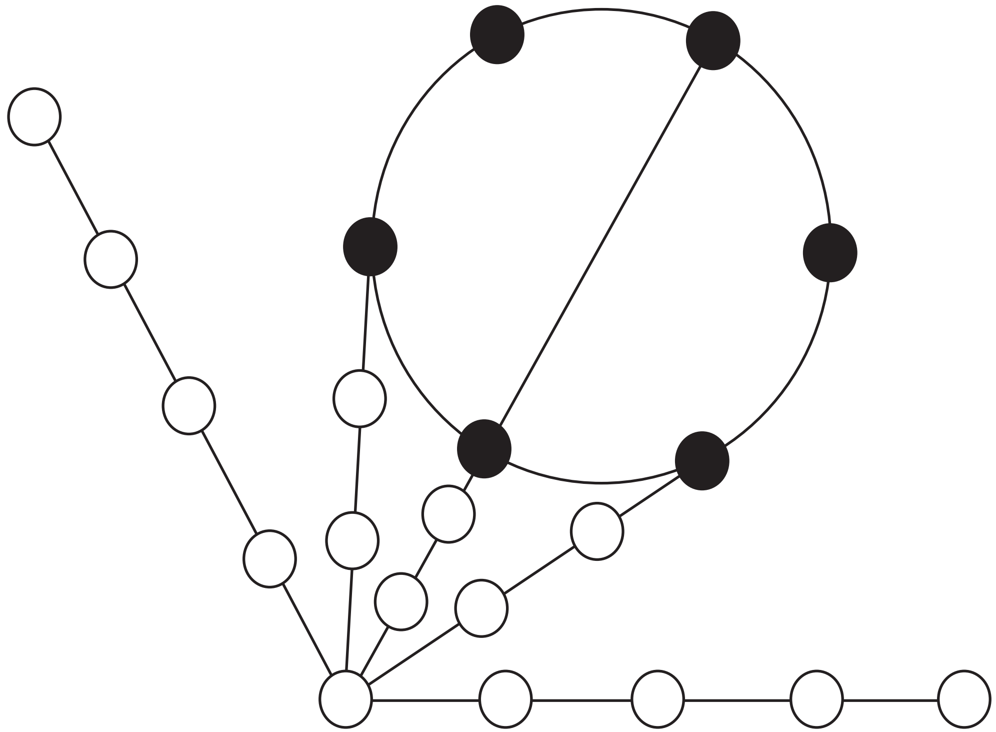
$\mathscr{P}$: $(\Phi_0; c_1, c_3, c_5) = (11; 5, 15, 1)$





$$\mathbf{V}(x_0^2 x_1^2 x_2^2 + x_0 x_1 x_2 + x_0^3).$$

$$\mathscr{O}^{c} \; : (\Phi_0; c_1, c_3, c_5) = (15; 0, 15, 6)$$



$$\mathbf{V}(x_0^2 x_1^2 x_2^2 + x_0 x_1 x_2).$$

Let Π_r be an r-flat PG(d, q).

Take Π_r and Π_s in PG(d, q) s.t. $\Pi_r \cap \Pi_s = \emptyset$.

For a set $\mathscr{K}$ in Π_r,

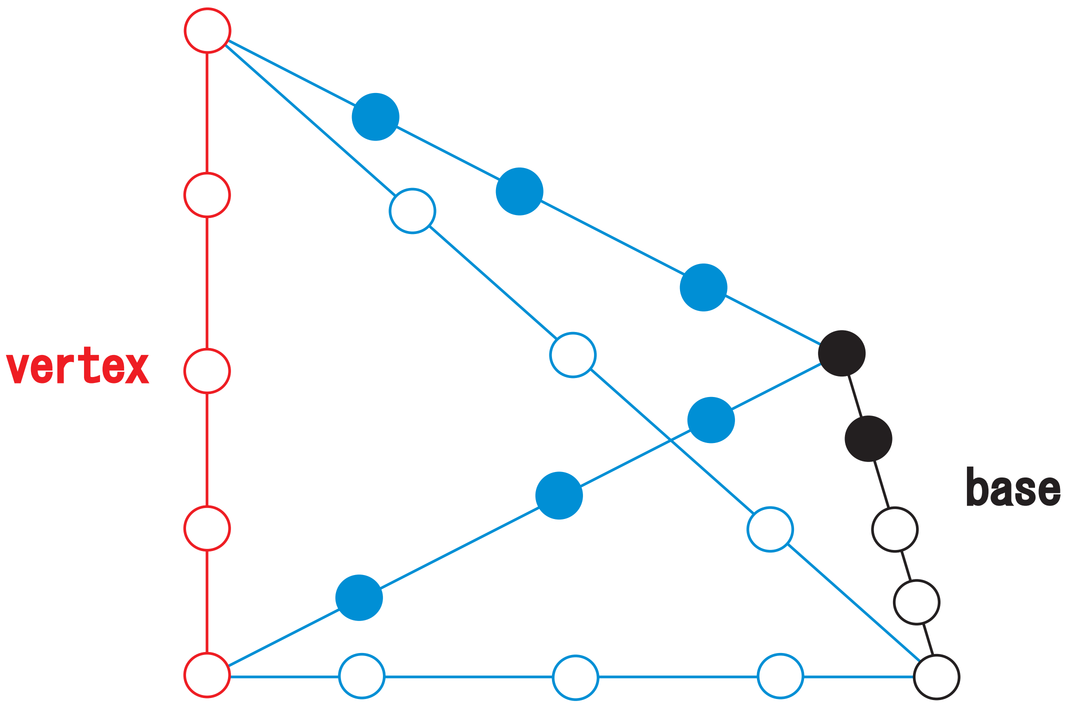
$$\Pi_s \mathscr{K} = \bigcup_{P \in \Pi_s, Q \in \mathscr{K}} \langle P, Q \rangle$$

is called a *cone* with *vertex* Π_s and *base* $\mathscr{K}$, where $\langle P, Q \rangle$ stands for the line through P and Q.

$\mathscr{K} \in O_r \Rightarrow \Pi_s \mathscr{K} \in O_d$

Example 1.

vertex Π_1 and *base* $\mathscr{U}_1 \longrightarrow (\Pi_1\mathscr{U}_1)$



For $\mathscr{K} \in O_d$ and a hyperplane π of Π_d, define the map $\delta_\pi : O_d \to O_d$ by $\mathscr{K}\delta_\pi = \mathscr{K}\nabla\pi$, with

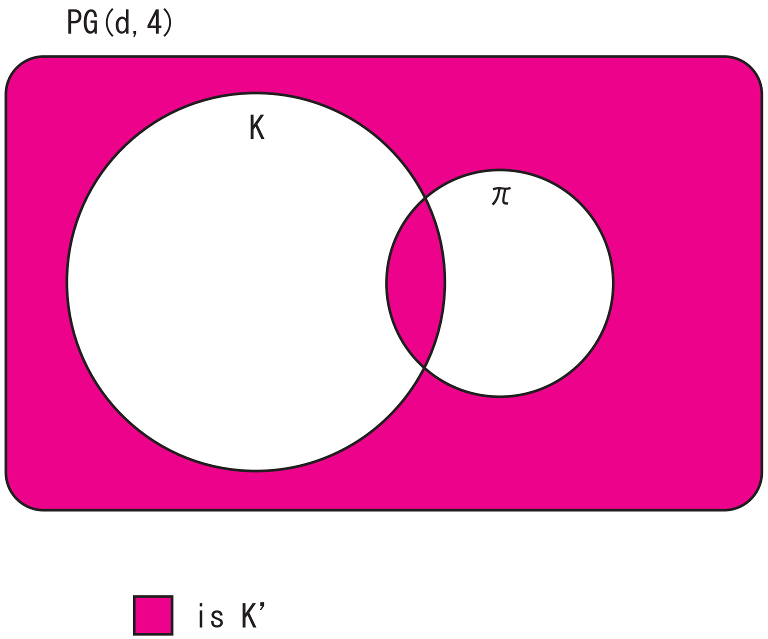
$$\mathscr{K}\nabla\pi = (\mathscr{K}^c \cap \pi^c) \cup (\mathscr{K} \cap \pi),$$

where $\mathscr{K}^c = \Pi_d \setminus \mathscr{K}$.

The map δ_π is called a *disflection by* π.

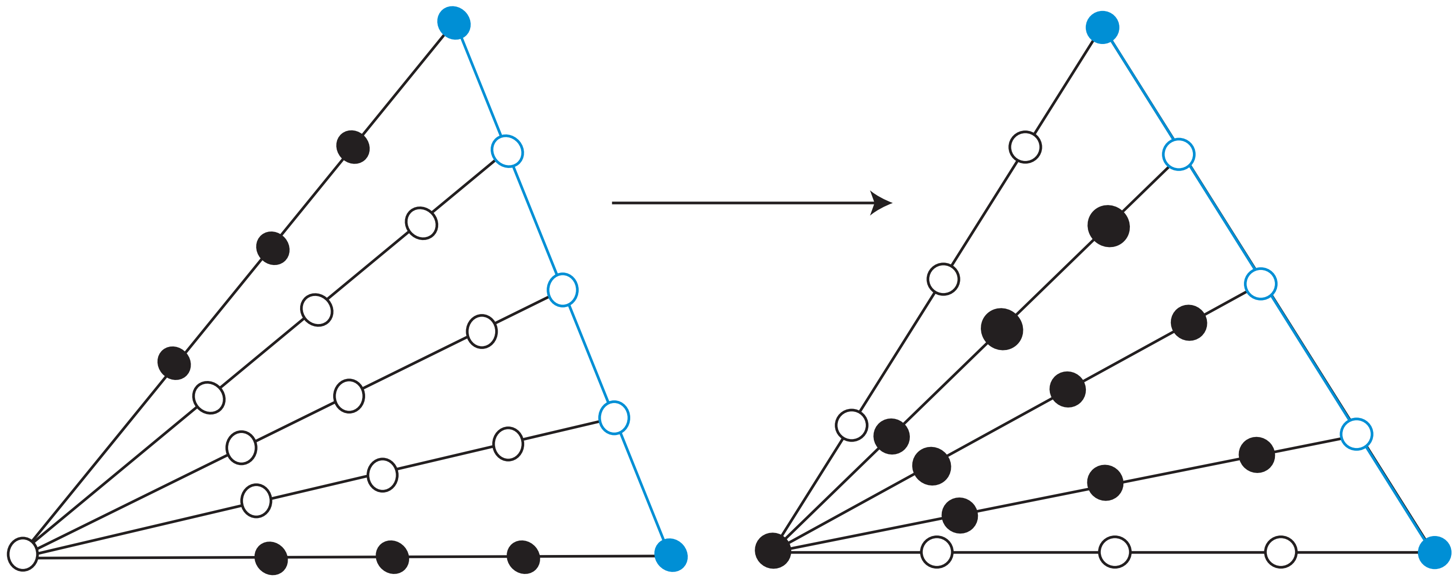
$|\mathscr{K}\nabla\mathscr{K}'| = |\Pi_d| - |\mathscr{K}| - |\mathscr{K}'| + 2|\mathscr{K} \cap \mathscr{K}'|$ for $\mathscr{K}, \mathscr{K}' \in O_d$.

$$\mathscr{K}' = \mathscr{K} \nabla \pi$$

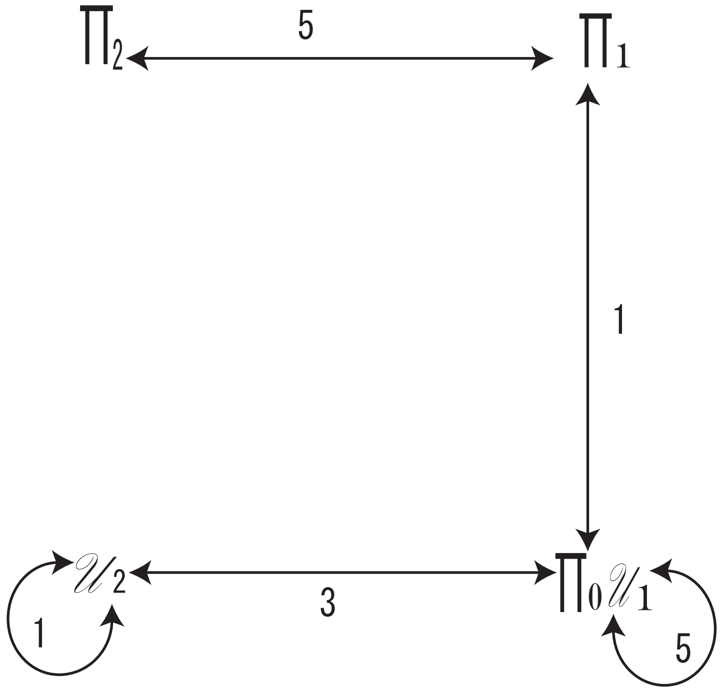


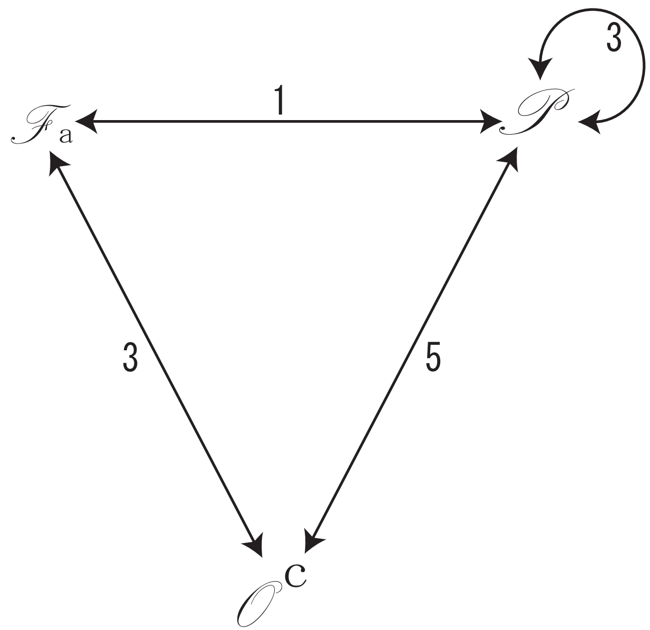
Example 2.

disflection ($\Pi_0\mathscr{U}_1 \to \mathscr{U}_2$)

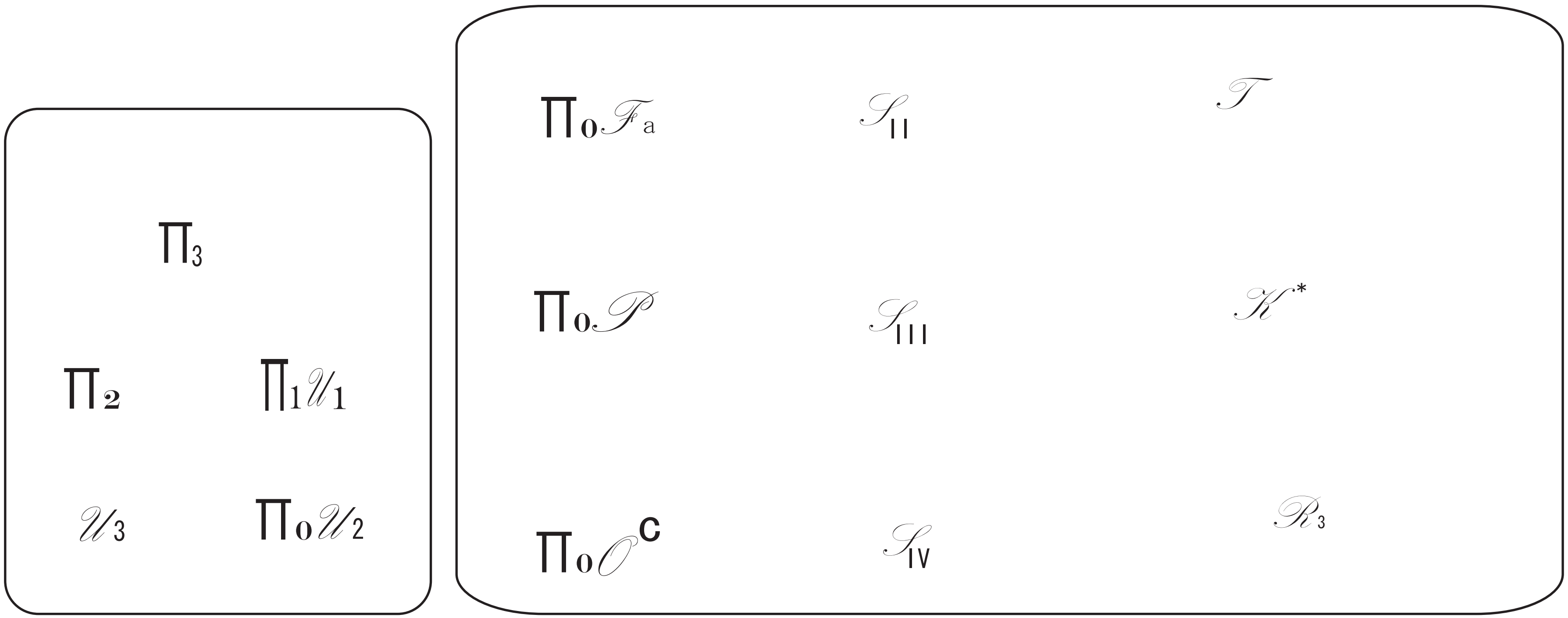


disflection diagram in $PG(2,4)$ [4]





disflection diagram in $PG(3,4)$ [4]



See Table 1 in the proceedings, P. 308.

No.1 $\sim$ 23 are found by
the cone construction and disflections.

No.1 $\sim$ 5 and No.7 $\sim$ 15 are obtaind by
the cone construction.
No.6 and No.16 $\sim$ 23 are found by
the disflection of them.

$\mathscr{K} \sim \mathscr{K}'$

if $\mathscr{K}$ and $\mathscr{K}'$ are projectively equivalent.

$\mathscr{K}$ and $\mathscr{K}'$ are of the same type.

Let

$$N(\mathscr{K}) := |\{\mathscr{K}' \in O_d \mid \mathscr{K} \sim \mathscr{K}'\}|.$$

Lem 6 (Sherman 1983)

The dimension of O_d

as a binary vector space is

$$\dim(O_d) = (d^3 + 3d^2 + 5d + 3)/3.$$

And

$$\sum N(\mathscr{K}) = |O_d| = 2^{\dim(O_d)}.$$

So $\sum N(\mathscr{K}) = |O_4| = 2^{45}$.

Lem 7.

Take Π_{d-s-1} and Π_s in $\mathrm{PG}(d,4)$ so that $\Pi_{d-s-1} \cap \Pi_s = \emptyset$. For a non-singular odd set $\mathscr{K}$ in Π_{d-s-1}, it holds that

$$N(\Pi_s \mathscr{K}) = N(\mathscr{K}) \times \frac{\theta_d \theta_{d-s-1}}{\theta_s}.$$

Non-singular odd sets don't have singular point. A point of $\mathscr{K}$ is singular if there is no 3-line through it.

Lem 8.

For an odd set $\mathscr{K}$ in Π_d

and a hyperplane Δ of Π_d,

let $\mathscr{K}' = \mathscr{K} \nabla \Delta$,

$s = |\{\pi \in \mathcal{F}_{d-1} \mid \mathscr{K} \nabla \pi \sim \mathscr{K}'\}|$, and

$s' = |\{\pi' \in \mathcal{F}_{d-1} \mid \mathscr{K}' \nabla \pi' \sim \mathscr{K}\}|$.

Then

$$N(\mathscr{K}') = N(\mathscr{K}) \times \frac{s}{s'}.$$

See Table 3, P. 310.

Then

$$\sum_{i=1}^{23} N(\mathscr{K}_i) < 2^{45}.$$

$\mathscr{K}_i$: the i-th odd set in Table 3.

In order to find a new odd set,

we use the next theorem.

Thm 9 (Sherman 1983)

Every odd set in PG$(d, 4)$ is uniquely expressed as

$$\mathrm{V}(E^2 + E + H),$$

where $E = \sum_{0<i,j,k<d} c_{ijk} x_i x_j x_k$

and H is Hermitian.

We found

$$\mathscr{V} : (\Phi_0; c_{85}, c_{61}, c_{53}) = (221; 1, 85, 255).$$

$\mathscr{V} = \mathrm{V}(E^2 + E)$,

where $E = x_0x_3x_4 + x_1x_2x_4$.

See Table 2 and Table 3, P. 309-310.

We found $\mathscr{V}$ and the disflected sets.

$$\sum_{i=1}^{45} N(\mathscr{K}_i) = 2^{45}.$$

Results

1. By the method of Brian Scherman, we found all odd sets in $PG(4,4)$.
2. They are classified to three cycles by disflection.
3. We give a new extension theorem as an application.

Thank you for your attention!

References

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[5] T. Maruta, Extendability of linear codes over $\mathbb{F}_q$, Proc. 11th International Workshop on Algebraic and Combinatorial Coding Theory (ACCT), Pamporovo, Bulgaria, 2008, 203–209.

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Lemma.

$N(\mathscr{V}) = N(\mathscr{P}_4) = 263983104.$

$\mathscr{P}_4$ is a parabolic quadric.

For $(x_0, x_1, x_2, x_3, x_4) \in PG(4,4)$

$\mathscr{P}_4 = \mathrm{V}(x_0x_3 + x_1x_2 + x_4^2).$