

On the types of book spreads of $PG(7, 2)$

Ronald Shaw Svetlana Topalova

University of Hull, Hull HU6 7RX, UK

Institute of Mathematics and Informatics, BAS

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- Introduction
- $(7, 2, 5, 3)$ book spreads
- Construction method
- Classification Results

Definition

A *t-spread* in $PG(n, q)$ is a partition of the points of the projective space by *t*-flats (*t*-dimensional subspaces).

Usually 1-spreads are called line spreads, or just spreads.

Spreads in projective spaces are used to construct constant dimension codes and spread codes.




-  T. Etzion and A. Vardy, Error-correcting codes in projective space, *IEEE Trans. Inform. Theory* 57 (2), 1165 - 1173, 2011.
-  N. Silberstein, T. Etzion, Codes and designs related to lifted MRD codes, Proceedings of *IEEE International Symposium on Information Theory (ISIT)*, St. Petersburg, 2288 - 2292, 2011 .
-  F. Manganiello, E. Gorla, and J. Rosenthal, Spread codes and spread decoding in network coding, in proceedings of *International Symposium on Information Theory*, Toronto, Ontario, Canada, 881 - 885, 2008.

Constructions of spreads



N. L. Johnson, *Combinatorics of Spreads and Parallelisms*,
Series: Chapman & Hall Pure and Applied Mathematics,
CRC Press, 2010.

Spreads in $PG(3, q)$ with certain automorphisms

-  V. Jha, N. L. Johnson, The classification of spreads in $PG(3, q)$ admitting linear groups of order $q(q + 1)$, II. Even order, *Advances in Geometry*, Special Issue, 271 – 313, 2003.
-  V. Jha, N. L. Johnson, The classification of spreads in $PG(3, q)$ admitting linear groups of order $q(q + 1)$, I. Odd order, *J. Geometry* 81, 46 – 80, 2004.
-  N. L. Johnson, Spreads in $PG(3, q)$ admitting several homology groups of order $q + 1$, *Note di Matematica* 24 (2), 9 – 39, 2005.

Maximal partial spreads in $PG(3, 2)$, $PG(3, 3)$, $PG(3, 4)$,
 $PG(4, 2)$




L. Soicher, Computation of Partial Spreads, web preprint,
<http://www.maths.qmul.ac.uk/~leonard/partialspreads>,
2000.




N. A. Gordon, R. Shaw and L. H. Soicher, Classification of
partial spreads in $PG(4, 2)$, 63 pp., Research Report, 2004.

Maximal partial spreads of size 45 in $PG(3, 7)$

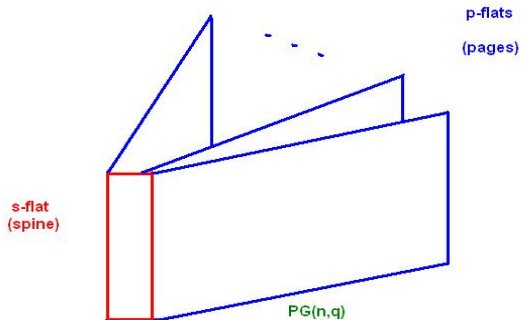
-  A. Blokhuis, A. E. Brouwer, H. A. Wilbrink, Blocking sets in $PG(2,p)$ for small p , and partial spreads in $PG(3,7)$, *Advances in Geometry*, Special Issue, 245 – 253, 2003.

All spreads in $PG(5, 2)$

-  Z. Mateva, S. Topalova. Line spreads of $PG(5, 2)$, *J. Combin. Des.* 17, 90 – 102, 2009.

Introduction - Book Spreads

(n, q, p, s) book



(n, q, p, s) book - a collection of p -flats of $PG(n, q)$ (*pages*), which cover the whole projective space and intersect in an s -flat (*spine*). Any point outside the spine is in exactly one page.

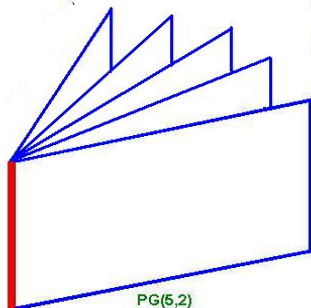
(n, q, p, s) book t -spread - a t -spread such that the points of each page of an (n, q, p, s) book and the points of the spine are partitioned by t -flats of this t -spread.

Introduction - Book Spreads in $PG(5, 2)$ I

(5,2,3,1) book spreads
(21 lines)

five pages
3-flats
(of 5 lines)

spine
a line



Ronald Shaw - idea about book spreads in $PG(5, 2)$ and classification without computer (2004)



T.P. McDonough, R. Shaw, S. Topalova, Classification of book spreads in $PG(5, 2)$, *Advances in Geometry*, under revision.

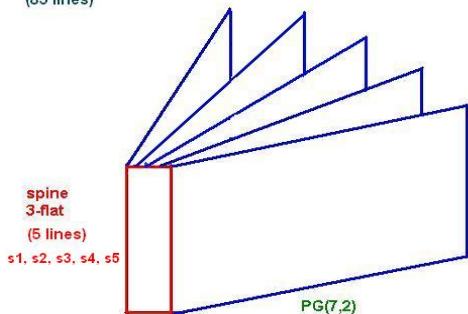
- rich automorphism groups
- 131044 inequivalent spreads
 - $|Aut| \geq 36$ - 26 spreads, 9 book spreads
 - $|Aut| \geq 72$ - 16 spreads, 8 book spreads
- no other spreads which partition at least five 3-flats

(7, 2, 5, 3) book spreads - Types

(7,2,5,3) book spreads
(85 lines)

P1, P2, P3, P4, P5

five pages
5-flats
(of 21 lines)



Three types of (7, 2, 5, 3) book spreads

- $(7, 2, 5, 3)_1$: The spread lines in each page form a (5, 2, 3, 1) book spread with spine s_1
- $(7, 2, 5, 3)_2$: The spread lines in page P_i form a (5, 2, 3, 1) book spread with spine s_i for $i = 1, 2, 3, 4, 5$.
- $(7, 2, 5, 3)_3$: all the remaining (7, 2, 5, 3) book spreads.

$(7, 2, 5, 3)_{1(2)}$ book spreads - 9 page types

Table: The nine $(5, 2, 3, 1)$ book spreads define nine page types of $(7, 2, 5, 3)_1$ and $(7, 2, 5, 3)_2$ book spreads, n_i - number of 3-flats containing i spread lines

page type	$ Aut $	n_3	n_4	n_5
1	362880	0	0	21
2	1728	0	12	9
3	1152	0	16	5
4	108	18	6	6
5	72	24	4	5
6	384	32	0	5
7	36	24	3	5
8	288	24	0	5
9	5760	0	0	5

additional restriction: the 5 pages should be of one and the same type.

Construction method - $(7, 2, 5, 3)_1$ book spreads

- We use our own C++ programs.
- We assign the spread lines numbers from 1 to 85 and construct spreads with the following properties:

$(7, 2, 5, 3)_1$ book spreads:

spine lines: 1, 2, 3, 4, 5

P_1 lines: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21

P_2 lines: 1, 2, 3, 4, 5, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37

P_3 lines: 1, 2, 3, 4, 5, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53

P_4 lines: 1, 2, 3, 4, 5, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69

P_5 lines: 1, 2, 3, 4, 5, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85

Table: $(5, 2, 3, 1)$ pages:

	P_1	P_2	P_3	P_4	P_5
p_1	1 2 3 4 5	1 2 3 4 5	1 2 3 4 5	1 2 3 4 5	1 2 3 4 5
p_2	1 6 7 8 9	1 22 23 24 25	1 38 39 40 41	1 54 55 56 57	1 70 71 72 73
p_3	1 10 11 12 13	1 26 27 28 29	1 42 43 44 45	1 58 59 60 61	1 74 75 76 77
p_4	1 14 15 16 17	1 30 31 32 33	1 46 47 48 49	1 62 63 64 65	1 78 79 80 81
p_5	1 18 19 20 21	1 34 35 36 37	1 50 51 52 53	1 66 67 68 69	1 82 83 84 85

Construction method - $(7, 2, 5, 3)_2$ book spreads

$(7, 2, 5, 3)_2$ book spreads:

spine lines: 1, 2, 3, 4, 5

P_1 lines: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21

P_2 lines: 1, 2, 3, 4, 5, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37

P_3 lines: 1, 2, 3, 4, 5, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53

P_4 lines: 1, 2, 3, 4, 5, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69

P_5 lines: 1, 2, 3, 4, 5, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85

Table: $(5, 2, 3, 1)$ pages:

	P_1	P_2	P_3	P_4	P_5
p_1	1 2 3 4 5	1 2 3 4 5	1 2 3 4 5	1 2 3 4 5	1 2 3 4 5
p_2	1 6 7 8 9	2 22 23 24 25	3 38 39 40 41	4 54 55 56 57	5 70 71 72 73
p_3	1 10 11 12 13	2 26 27 28 29	3 42 43 44 45	4 58 59 60 61	5 74 75 76 77
p_4	1 14 15 16 17	2 30 31 32 33	3 46 47 48 49	4 62 63 64 65	5 78 79 80 81
p_5	1 18 19 20 21	2 34 35 36 37	3 50 51 52 53	4 66 67 68 69	5 82 83 84 85

Construction method - backtrack search

- we fix the **first 6** lines and line **22**.
- a set D of **1747** lines (out of **all 10795** lines of $PG(7, 2)$) that are skew to each of the fixed 7 ones.
- **backtrack search** to choose the other spread lines from D
- **isomorphism test** on the partial solutions **after a whole page**, namely after adding lines 21, 37, 53, 69 and 85.

Construction method - isomorphism test

Before starting the search

- we find all 108 automorphisms which stabilize the fixed lines.
- we find for each line of D automorphisms mapping it to:
 - line 1,
 - line 2 and fixing line 1,
 - line 6 and fixing lines 1, 2, ..., 5,
 - line 22 and fixing lines 1, 2, ..., 21.

Isomorphism test - test of the current solution for minimality

- by the above listed automorphisms we map the spread lines to the 7 fixed lines in all possible ways.
- in each case we apply the 108 automorphisms, which stabilize the fixed lines.
- If one of these automorphisms maps the current solution to a lexicographically smaller one, we drop the current partial solution.

Classification Results I

Table: $(7, 2, 5, 3)_1 / (7, 2, 5, 3)_2$ book spreads with all pages of type 1

$ Aut $	spreads
5922201600	1
1105920	1
221184	1
207360	1
73728	1
18432	1
15360	1

$ Aut $	spreads
13824	1
9216	1
6480	1
3456	1
1152	1
432	1
All	13

Classification Results II

Table: $(7, 2, 5, 3)_1$ book spreads with all pages of type 2

Aut	spreads
1	57823
2	8624
3	1344
4	655
6	2027
8	264
9	31
12	305
16	14
18	445
24	236
32	10
36	87
48	35
54	74

Aut	spreads
72	95
96	60
108	23
120	1
144	6
162	5
192	2
216	22
288	37
324	4
360	3
384	10
432	2
576	1
648	3

Aut	spreads
864	13
1080	2
1152	12
1296	1
1440	1
3456	7
4320	1
4608	1
5184	1
13824	2
20736	1
41472	1
69120	1
All	72292

Classification Results III

Table: $(7, 2, 5, 3)_1$ book spreads with all pages of type 3

Aut	spreads
1	6769
2	395
3	153
4	25771
6	34
8	1387
12	1267
16	17811
24	249
32	1259
36	16
48	3115
64	176
72	14

Aut	spreads
96	297
128	61
144	72
192	2531
256	7
288	14
320	1
384	365
512	3
576	51
768	66
960	1
1152	39
1536	24

Aut	spreads
1920	1
2304	13
3072	3
3840	2
4608	6
5760	1
6144	2
9216	1
11520	1
18432	2
27648	1
55296	1
73728	1
184320	1
All	61984

Classification Results IV

Table: $(7, 2, 5, 3)_1$ book spreads with all pages of type 9

$ Aut $	spreads
1	1
2	2
4	5
8	4
12	2
16	2
24	2
32	4
48	13
64	6

$ Aut $	spreads
96	8
128	4
144	1
192	5
256	2
384	6
512	2
768	2
960	1
1024	2

$ Aut $	spreads
1152	1
1536	3
2048	1
3072	2
3840	1
4608	1
6144	1
11520	1
12288	1
18432	1
All	87

Classification Results V

Table: $(7, 2, 5, 3)_2$ book spreads with all pages of type 9

$ Aut $	spreads
1	86
2	26
3	17
4	19
6	14
8	15
9	1
12	7
16	4
18	2

$ Aut $	spreads
24	5
32	6
36	2
72	2
80	1
96	2
240	1
360	1
All	211

Thank you

Thank you for the attention!