Classification of the (12,19,1,2) and (12,20,1,2) superimposed codes

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Definition 1

A binary $N \times T$ *matrix* C *is called an* (N, T, w, r) *superimposed code (SIC) of length* N *and size* T *if for any pair of subsets* $W, R \subset \{1, 2, \ldots, T\}$ *such that* $|W| = w$, $|R| = r$ *and* $W \cap R = \varnothing$, *there exists a row* $i \in \{1, 2, \ldots, N\}$ *such that* $c_{ij} = 1$ *for all* $j \in W$ *and* $c_{ij} = 0$ *for all* $j \in R$.

Introduction

Example: $(9, 12, 1, 2)$ SIC

0	0	0	0	0	0	0	0	1	1	1	1
0	0	0	0	0	1	1	1	0	0	0	1
0	0	0	1	1	0	0	1	0	0	1	0
0	0	1	0	1	0	1	0	0	1	0	0
0	1	0	0	1	1	0	0	1	0	0	0
0	1	1	1	0	0	0	0	0	0	0	1
1	0	0	1	0	0	1	0	1	0	0	0
1	0	1	0	0	1	0	0	0	0	1	0
1	1	0	0	0	0	0	1	0	1	0	0

Introduction

$W = \{1\} \quad R = \{2, 3\}$

0	0	0	0	0	0	0	0	1	1	1	1
0	0	0	0	0	1	1	1	0	0	0	1
0	0	0	1	1	0	0	1	0	0	1	0
0	0	1	0	1	0	1	0	0	1	0	0
0	1	0	0	1	1	0	0	1	0	0	0
0	1	1	1	0	0	0	0	0	0	0	1
1	0	0	1	0	0	1	0	1	0	0	0
1	0	1	0	0	1	0	0	0	0	1	0
1	1	0	0	0	0	0	1	0	1	0	0

Introduction

$W = \{1\}$ $R = \{2, 3\}$ $i = 7$

0	0	0	0	0	0	0	0	1	1	1	1
0	0	0	0	0	1	1	1	0	0	0	1
0	0	0	1	1	0	0	1	0	0	1	0
0	0	1	0	1	0	1	0	0	1	0	0
0	1	0	0	1	1	0	0	1	0	0	0
0	1	1	1	0	0	0	0	0	0	0	1
1	0	0	1	0	0	1	0	1	0	0	0
1	0	1	0	0	1	0	0	0	0	1	0
1	1	0	0	0	0	0	1	0	1	0	0

$$W = \{3\} \qquad R = \{2, 4\}$$

0	0	0	0	0	0	0	0	1	1	1	1
0	0	0	0	0	1	1	1	0	0	0	1
0	0	0	1	1	0	0	1	0	0	1	0
0	0	1	0	1	0	1	0	0	1	0	0
0	1	0	0	1	1	0	0	1	0	0	0
0	1	1	1	0	0	0	0	0	0	0	1
1	0	0	1	0	0	1	0	1	0	0	0
1	0	1	0	0	1	0	0	0	0	1	0
1	1	0	0	0	0	0	1	0	1	0	0

$W = \{3\}$ $R = \{2, 4\}$ $i = 4$ or 8

0	0	0	0	0	0	0	0	1	1	1	1
0	0	0	0	0	1	1	1	0	0	0	1
0	0	0	1	1	0	0	1	0	0	1	0
0	0	1	0	1	0	1	0	0	1	0	0
0	1	0	0	1	1	0	0	1	0	0	0
0	1	1	1	0	0	0	0	0	0	0	1
1	0	0	1	0	0	1	0	1	0	0	0
1	0	1	0	0	1	0	0	0	0	1	0
1	1	0	0	0	0	0	1	0	1	0	0

Introduction

The main problems in the theory of the superimposed codes:

- find the minimum length $N(T, w, r)$ for which an (N, T, w, r) SIC exists;
- find the maximum size $T(N, w, r)$ for which an (N, T, w, r) SIC exists.

Introduction

Values of $N(T, 1, 2)$ for $T \leq 21$

T	3	4	5	6	7	8	$9-12$	13	$14-17$	$18-20$	21
$N(T, 1, 2)$	3	4	5	6	7	8	9	10	11	12	≥ 12

Introduction

Definition 2 *Two* (N, T, w, r) *superimposed codes are equivalent if one of them can be transformed into the other by a permutation of the rows and a permutation of the columns.*

Introduction

Number of nonequivalent classes of optimal $(N(T,1,2), T, 1, 2)$ superimposed codes for $T \leq 17$

T	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
#	1	1	1	1	2	4	25	4	1	1	5	2705	278	21	2

Number of nonequivalent classes of optimal $(N(T,1,2),T,1,2)$ superimposed codes for $T \leq 20$

T	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
#	1	1	1	1	2	4	25	4	1	1	5	2705	278	21	2

T	18	19	20
#	?	594	5

Introduction

Values of $N(T,1,2)$ for $T \leq 21$

T	3	4	5	6	7	8	$9-12$	13	$14-17$	$18-20$	21
$N(T,1,2)$	3	4	5	6	7	8	9	10	11	12	13

Introduction

The results have been obtained using:

- an exhaustive computer search for the generation of $(N, T, 1, 1)$ and $(N, T, 1, 2)$ superimposed codes;
- the program *Q-extension* (Bouyukliev) for code equivalence testing.

Preliminaries

Theorem 3 *(Sperner Theorem)* $T(N,1,1) = \binom{N}{\lfloor N/2 \rfloor}$.

Preliminaries

Definition 4 *The residual code* $Res(C, x = v)$ *of a superimposed code* C *with respect to value* v *in column* x *is a code obtained by taking all the rows in which* C *has value* v *in column* x *and deleting the* x^{th} *entry in the selected rows.*

Preliminaries

$$
C = \begin{array}{cccccc}
x & & & & & \\
0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0
\end{array}
$$

Preliminaries

$$C = \begin{array}{cccccc} x & & & & & \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{array}$$

(Rows 1–10, columns 2–6) $= Res(C, x = 0)$

Preliminaries

$$C = \begin{array}{cccccc} x & & & & & \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{array}$$

Rows 1–10 (columns 2–6) $= Res(C, x = 0)$

Rows 11–15 (columns 2–6) $= Res(C, x = 1)$

Preliminaries

S_x – the characteristic set of column x.

L_p – the characteristic set of row p.

Lemma 5 *Let C be an $(N, T, 1, 2)$ superimposed code and x be a column of C. Then $Res(C, x = 0)$ is an $(N - |S_x|, T - 1, 1, 1)$ superimposed code.*

Lemma 6 *Let C be an (N, T, w, r) superimposed code and x be a column of C. The matrix $C' = C \backslash \{x\}$ is an $(N, T - 1, w, r)$ superimposed code.*

Preliminaries

Lemma 7 *Suppose C is an $(N, T, 1, 2)$ superimposed code and x is a column such that $|S_x| \leq 2$. Then there exists a row p for which $c_{px} = 1$ and $|L_p| = 1$.*

Lemma 8 *Suppose C is an $(N, T, 1, 2)$ superimposed code and p is a row such that $|L_p| = 1$. Then there exists an $(N-1, T-1, 1, 2)$ superimposed code.*

Lemma 9 $N(T-1, 1, 2) \leq N(T, 1, 2) \leq N(T-1, 1, 2) + 1$.

Classification of the (12,19,1,2) superimposed codes

Lemma 10 *Suppose C is a $(12, 19, 1, 2)$ superimposed code and x and y are two different columns of C. Then $|S_x \cap \overline{S_y}| \geq 2$.*

Lemma 11 *Suppose C is a $(12, 19, 1, 2)$ superimposed code and x is a column of C. Then $3 \leq |S_x| \leq 6$.*

Classification of the (12,19,1,2) superimposed codes

Lemma 12 *Let C be a $(12, 19, 1, 2)$ superimposed code. Then there is no column x of C for which $|S_x| = 6$.*

Classification of the (12,19,1,2) superimposed codes

Lemma 12 *Let C be a $(12,19,1,2)$ superimposed code. Then there is no column x of C for which $|S_x| = 6$.*

Proof

$$\left(\begin{array}{c|c} 0 & \\ \vdots & C_0 \\ 0 & \\ \hline 1 & \\ \vdots & X \\ 1 & \end{array}\right)$$

- C_0 is $(6, 18, 1, 1)$ SIC;
- the rows and the columns of the matrix C_0 are sorted lexicographically;
- the rows of the matrix X are sorted lexicographically;
- all columns of C have weight between 3 and 6;
- $|S_y \cap \overline{S_z}| \geq 2$ for every two columns y and z of C.

Classification of the (12,19,1,2) superimposed codes

Lemma 12 *Let C be a $(12, 19, 1, 2)$ superimposed code. Then there is no column x of C for which $|S_x| = 6$.*

Proof

$$\left(\begin{array}{c|c} 0 & \\ \vdots & C_0 \\ 0 & \\ \hline 1 & \\ \vdots & X \\ 1 & \end{array}\right)$$

- C_0 is $(6, 18, 1, 1)$ SIC;
- the rows and the columns of the matrix C_0 are sorted lexicographically;
- the rows of the matrix X are sorted lexicographically;
- all columns of C have weight between 3 and 6;
- $|S_y \cap \overline{S_z}| \geq 2$ for every two columns y and z of C.

There are exactly 3 inequivalent possibilities for C_0.
The extension is impossible.

Classification of the (12,19,1,2) superimposed codes

Lemma 13 *Let C be a $(12, 19, 1, 2)$ superimposed code. Then there is a row p of C for which $|L_p| \leq 7$.*

Classification of the (12,19,1,2) superimposed codes

Theorem 14 *There are exactly* 594 *inequivalent* $(12, 19, 1, 2)$ *superimposed codes.*

Classification of the (12,19,1,2) superimposed codes

Theorem 14 *There are exactly* 594 *inequivalent* $(12, 19, 1, 2)$ *superimposed codes.*

Proof

$$\left(\begin{array}{cccccccccccc|ccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{1,13} & c_{1,14} & c_{1,15} & c_{1,16} & c_{1,17} & c_{1,18} & c_{1,19} \\ \hline & & & & & & & & & & & & & & & & & & \\ & & & & & A & & & & & & & & & & & & & \\ & & & & & & & & & & & & & & & & & & \end{array}\right)$$

- A is an $(11, 12, 1, 2)$ superimposed code;
- the rows and the columns of A are sorted lexicographically;
- the last 7 columns of C are sorted lexicographically;
- all columns of C have weight between 3 and 5;
- $|S_y \cap \overline{S_z}| \geq 2$ for every two columns y and z of C.

Classification of the (12,19,1,2) superimposed codes

Theorem 14 *There are exactly* 594 *inequivalent* $(12, 19, 1, 2)$ *superimposed codes.*

Proof

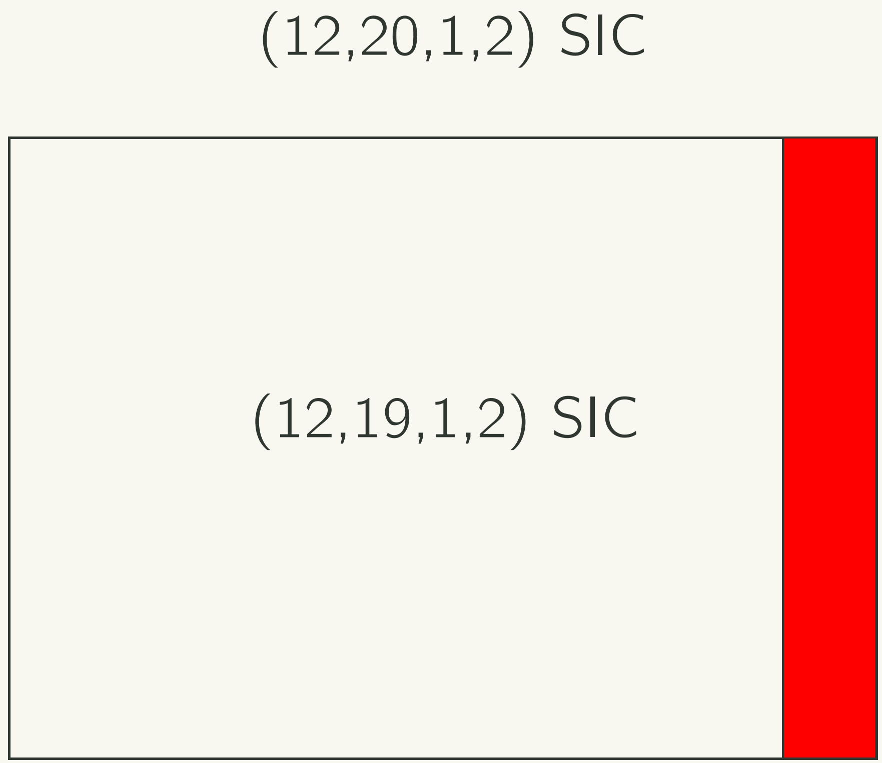
$$
\left(\begin{array}{cccccccccccc|ccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{1,13} & c_{1,14} & c_{1,15} & c_{1,16} & c_{1,17} & c_{1,18} & c_{1,19} \\
\hline
 & & & & & & & & & & & & & & & & & & \\
 & & & & & & A & & & & & & & & & & & & \\
 & & & & & & & & & & & & & & & & & &
\end{array}\right)
$$

- A is an $(11, 12, 1, 2)$ superimposed code;
- the rows and the columns of A are sorted lexicographically;
- the last 7 columns of C are sorted lexicographically;
- all columns of C have weight between 3 and 5;
- $|S_y \cap \overline{S_z}| \geq 2$ for every two columns y and z of C.

There are exactly 239232 inequivalent possibilities for A.

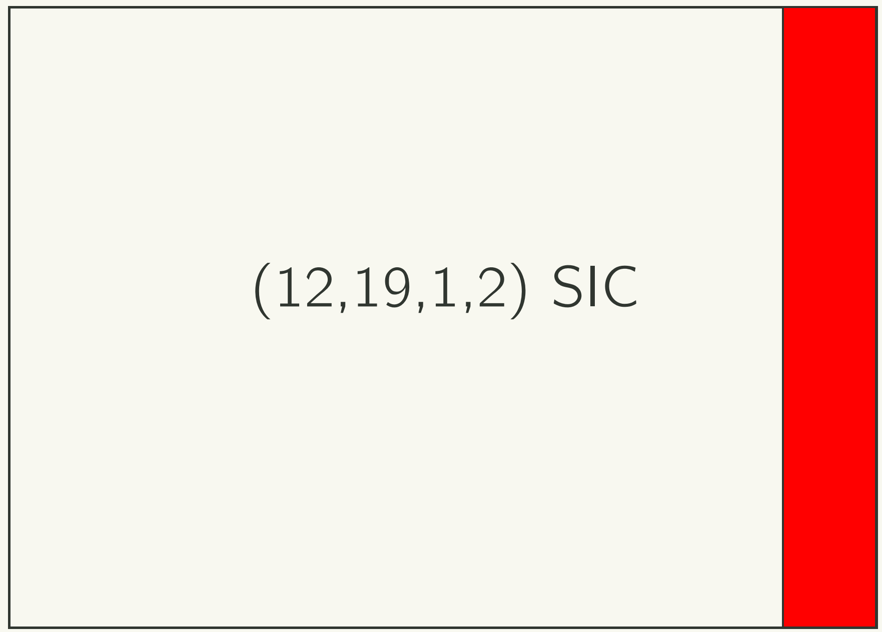
There are exactly 594 inequivalent $(12, 19, 1, 2)$ superimposed codes.

Classification of the (12,20,1,2) superimposed codes



Classification of the (12,20,1,2) superimposed codes

(12,20,1,2) SIC



Theorem 15 *There are exactly* 5 *inequivalent* $(12, 20, 1, 2)$ *superimposed codes.*

Classification of the (12,20,1,2) superimposed codes

The representatives of all inequivalent $(12, 20, 1, 2)$ superimposed codes

1	**2**	**3**
00000000000000011111	00000000000000011111	00000000000000011111
00000000000111100001	00000000000111100001	00000000000111100001
00000000111000100010	00000000111000100010	00000000111000100010
00000011001001000100	00000011001001000100	00000011001001000100
00000101010010001000	00000101010010001000	00000101010010001000
00001001100100010000	00001001100100010000	00001001100100010000
00110010000000101000	00110010000000110000	00110010000000110000
01010100000001010000	01010100000001000010	01010100000001000010
01101000000010000010	01101000000010000100	01101000001010000000
10011000010000000100	10011000010000000001	10011000010000000001
10100100001100000000	10100100001100000000	10100100000100000100
11000010100000000001	11000010100000001000	11000010100000001000

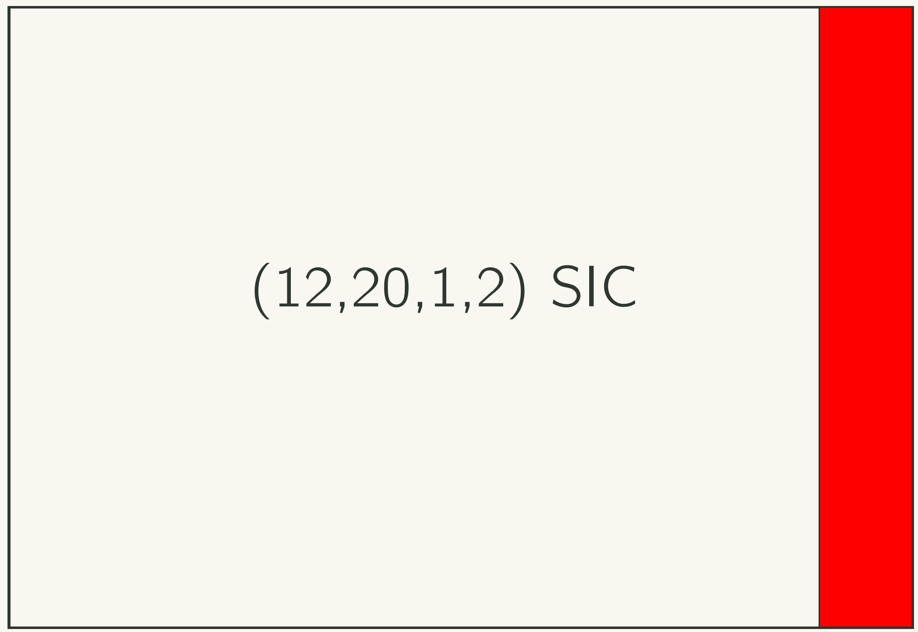
Classification of the (12,20,1,2) superimposed codes

The representatives of all inequivalent $(12, 20, 1, 2)$ superimposed codes

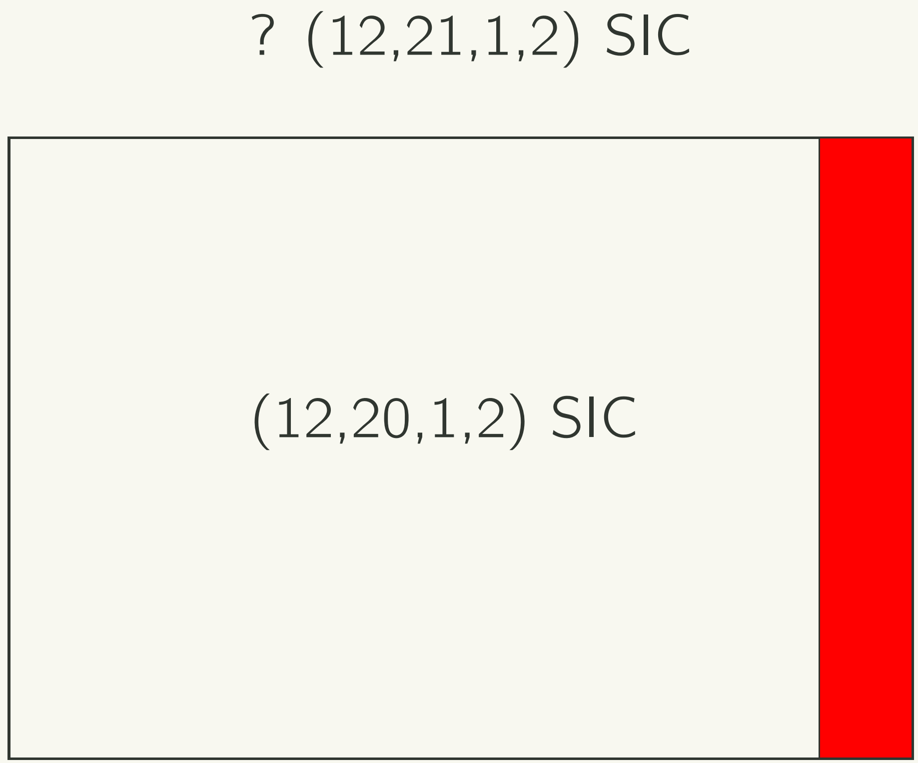
4	**5**
00000000000000011111	00000000000000011111
00000000000111100001	00000000000111100001
00000000111000100010	00000000111000100010
00000011001001000100	00000011001001000100
00000101010010001000	00000101010010001000
00001001100100010000	00001001100100010000
00110010000010000010	00110010000010000010
01010100000000110000	01010100000000110000
01101000001000000001	01101000001000001000
10011000010001000000	10011000010001000000
10100100000100000100	10100100000100000100
11000010100000001000	11000010100000000001

Classification of the (12,20,1,2) superimposed codes

? (12,21,1,2) SIC



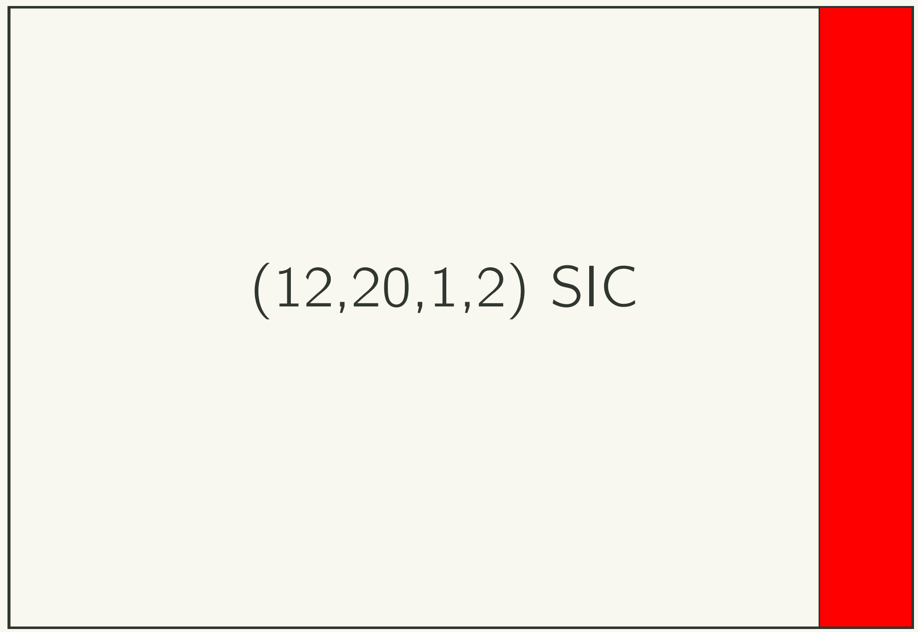
Classification of the (12,20,1,2) superimposed codes



There is no $(12, 21, 1, 2)$ SIC.

Classification of the (12,20,1,2) superimposed codes

? (12,21,1,2) SIC

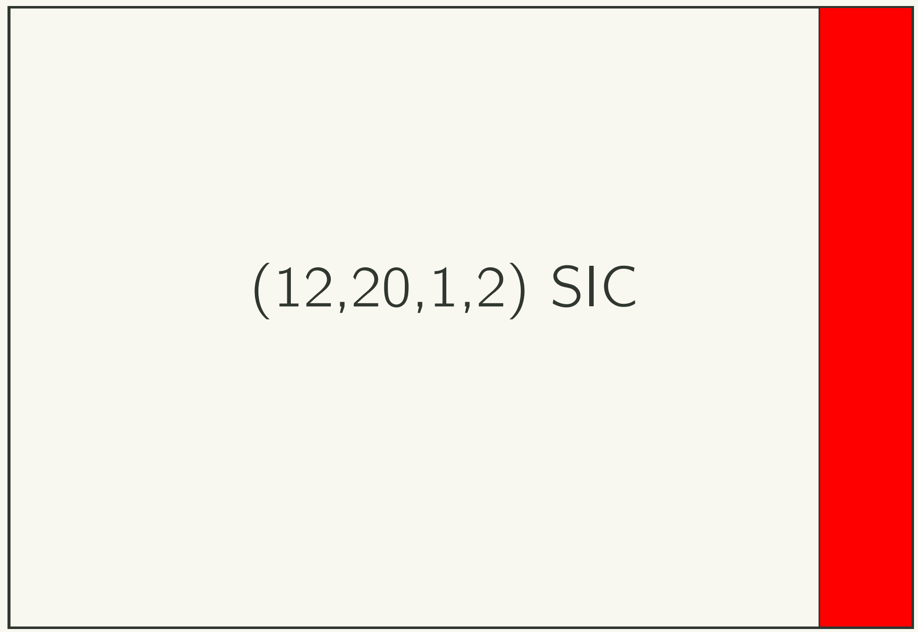


There is no $(12, 21, 1, 2)$ SIC.

Theorem 16 $T(12, 1, 2) = 20$.

Classification of the (12,20,1,2) superimposed codes

? (12,21,1,2) SIC



There is no $(12, 21, 1, 2)$ SIC.

Theorem 16 $T(12, 1, 2) = 20$.

Theorem 17 $N(21, 1, 2) = 13$.