ON THE SHARPNESS OF THE JAMISON-BRUEN BOUND

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1. Preliminaries

A **t-fold blocking set** with respect to hyperplanes in $\mathrm{AG}(n,q)$ is a set $\mathcal{B}$ of points such that each hyperpalne intersects it in at least t points. A t-fold blocking set of cardinality N is also referred to as an **(N,t)-blocking set**.

Theorem 1. The existence of the following objects is equivalent:

(1) an $[n,k,d]_q$ linear code with a word of maximal weight n;

(2) an $(n,n-d)$-arc in $\mathrm{PG}(k-1,q)$ with an empty hyperplane;

(3) an affine $(q^{k-1}-n, q^{k-2}-n+d)$-blocking set in $\mathrm{AG}(k-1,q)$.

2. Lower Bounds on the Size of a t-fold Blocking Set

- **Jamison (1977):** For $t = 1$

$$N \geq n(q-1) + 1$$

- **Brouwer, Schrijver (1978):** For $t = 1$ (shorter proof, polynomial method)

$$N \geq n(q-1) + 1$$

- **Bruen (1992):** For any $t \geq 1$

$$N \geq (n+t-1)(q-1) + 1.$$

- **Ball (2000):** For $t < q$

$$N \geq (n+t-1)(q-1)+k$$

provided there exists a j, $k-1 \leq j < t$, with $\binom{k-n-t}{j} \not\equiv 0 \pmod{p}$.

- **Ball (2000):** For $t < q$

$$N \geq (n+t-1)q - n + 1$$

provided $\binom{-n}{t-1} \not\equiv 0 \pmod{p}$.

- **Zanella (2002):** Bruen's bound cannot be attained for values of t with

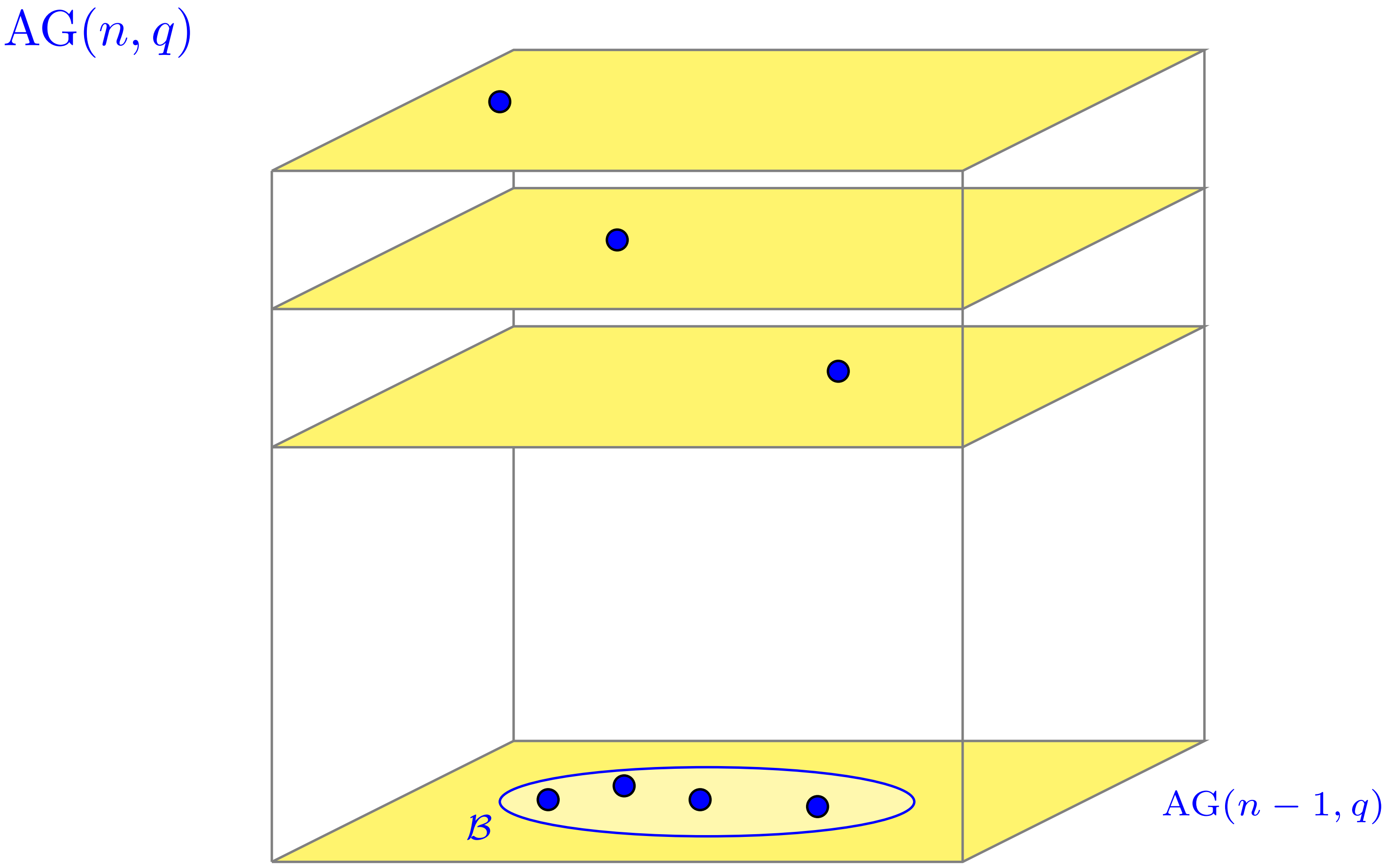
$$t > \frac{1}{2}(n-1)(q-1)+1.$$

3. Blocking Sets Meeting the Jamison-Bruen Bound

- **Brouwer, Schrijver (1978):** If $t = 1$ equality in Jamison's bound is achieved in all affine geometies $\mathrm{AG}(n, q)$.

- a plane affine blocking set meeting the Jamison-Bruen's bound trivially exists (e.g. two non-parallel lines);

- induction on the dimension n.



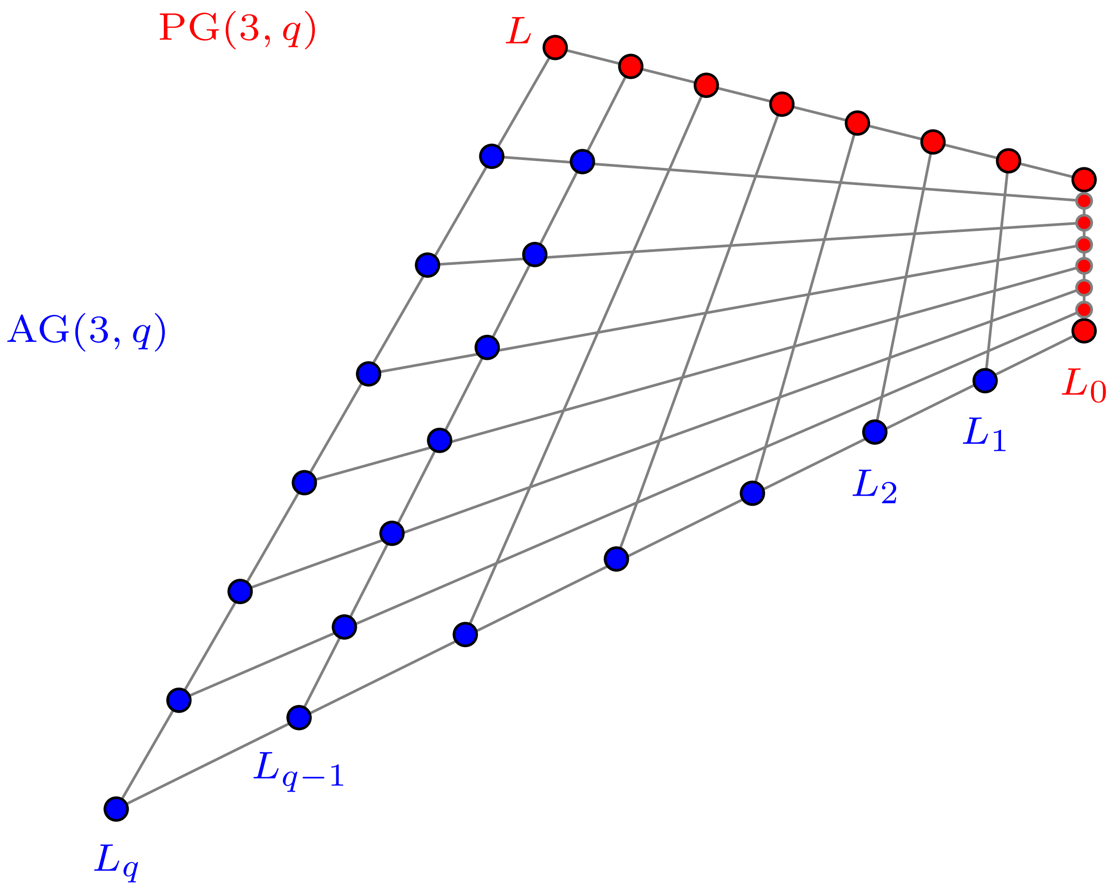
$\mathcal{B}$: $((n-1)(q-1)+1)$-blocking set

- **Ball (2000):** If $n = 3$ and $t = q-1$ equality in Jamison's bound is achieved in all affine geometies $\mathrm{AG}(3, q)$.

Theorem 2. There exists a $(q^2, q-1)$-blocking set in $\mathrm{AG}(3, q)$ for every prime power q.

- e.g. hyperbolic quadric with tangent plane at infinity

A $(q^2, q-1)$-blocking set in $\mathrm{AG}(3,q)$



Corollary 3. There exists a $(q^2-s(s+1), q-(s+1))$-blocking set in $\mathrm{AG}(3,q)$ for every prime power q and every $s=1,\ldots,q-2$.

- given a $(q^2, q-1)$-blocking set in $\mathrm{AG}(3,q)$, delete $s+1$ points from each of the lines $L_1,\ldots,L_s$.

Corollary 4. There exists a $(q^2-q, q-4)$-blocking set in $\mathrm{AG}(3,q)$ for every prime power q.

- given a $(q^2, q-1)$-blocking set in $\mathrm{AG}(3,q)$, delete **one** point from each of the lines $L_1,\ldots,L_q$ that are in general position.

- this construction is better for large q, $q \geq 13$.

• **Ball (2000):** If $t = 2$ equality in Ball's improvement of Jamison-Bruen bound is achieved in all affine geometies $\mathrm{AG}(n, q)$.

- $\mathcal{C} = \{P_1, P_2, \ldots, P_{n+1}\}$ - $(n+1)$-arc in $\Sigma_\infty = \mathrm{PG}(n-1, q)$;

- $L_1, L_2, \ldots, L_{n+1}$ - lines in $\mathrm{PG}(n, q)$, $P_i \in L_i$;

- $L_i \cap L_{i+1} = Q_{i,i+1}$, $L_{n+1} \cap L_1 = Q_{n+1,1}$, $L_i \cap L_j = \varnothing$ – otherwise;

- $H_{i,i+1} = \langle P_1, \ldots, P_{i-1}, Q_{i,i+1}, P_{i+1}, \ldots, P_{n+1} \rangle$, $i = 1, \ldots, n$;

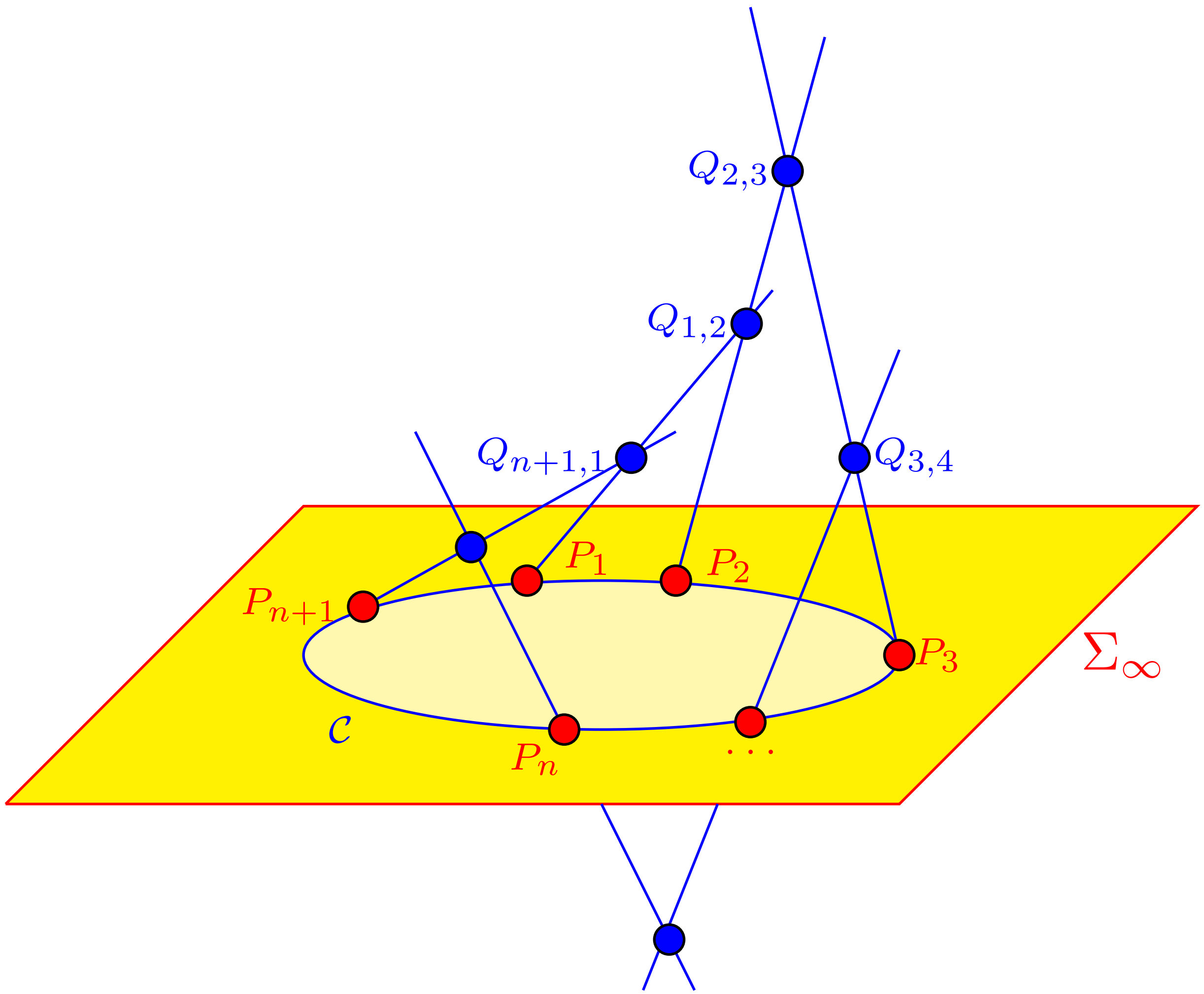
- $H_{n,n+1} = \langle P_2, \ldots, P_n, Q_{n+1,1} \rangle$;

- $\mathcal{B} = \cup L_i \setminus \Sigma_\infty$, $P = H_{12} \cap H_{23} \cap \ldots \cap H_{n,n+1}$, $Q \in H_{n+1,1}$;

- $\mathcal{B} \cup \{P, Q\}$ is a double affine blocking set;

- for $n \equiv 0 \pmod{p}$ one can make $P = Q$.

An $((n+1)q - n + \varepsilon, 2)$-blocking set in $\mathrm{AG}(n, q)$



Theorem 5. There exists a $(q^2, q-n+2)$-blocking set in $\mathrm{AG}(n,q)$ for every prime power q and every $3 \leq n \leq q+1$.

- T a subspace of codimension 2 in $\Omega = \mathrm{PG}(n,q)$;

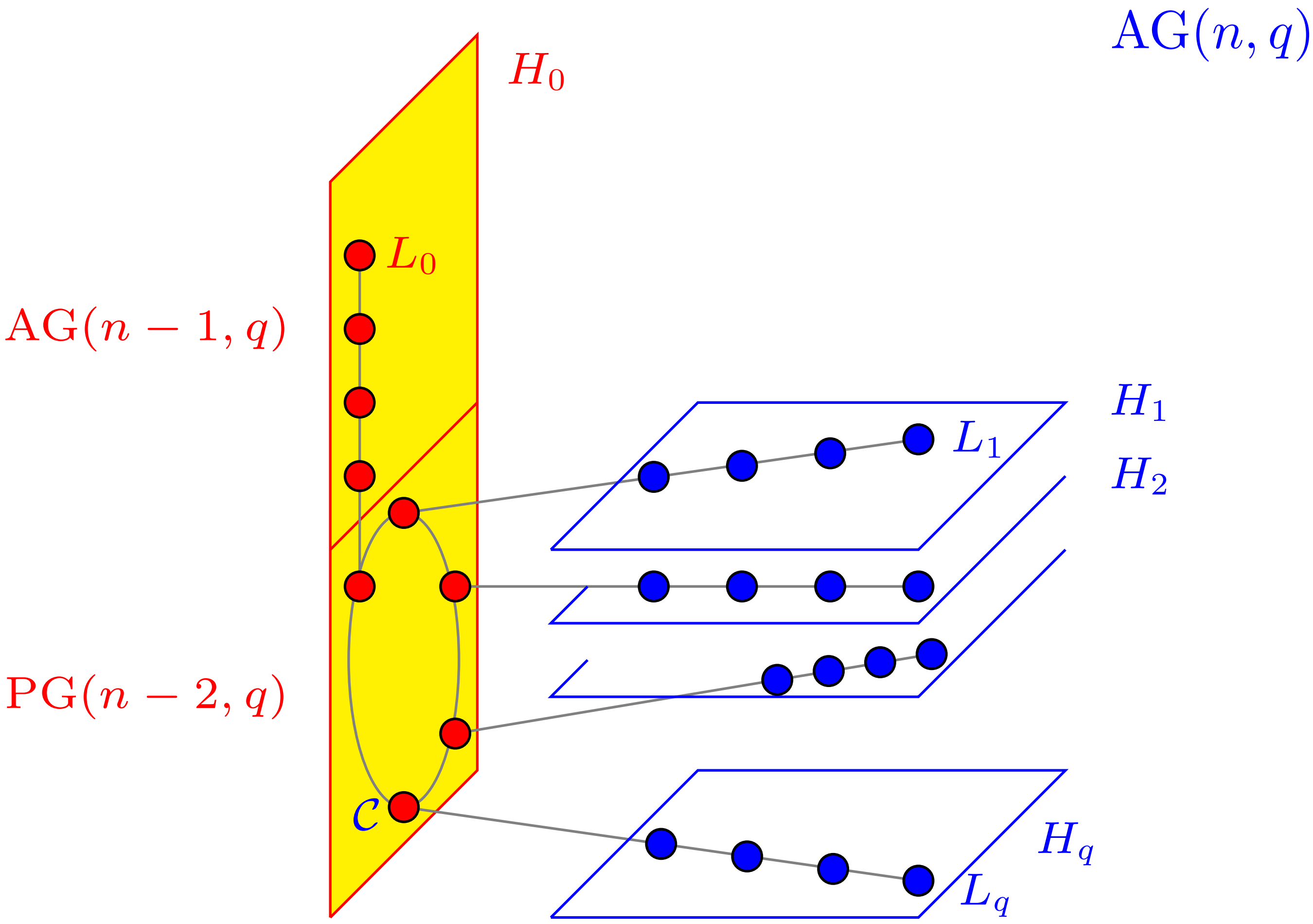
- $H_0, \ldots, H_q$ the hyperplanes through T in Ω.

- $\mathcal{C} = \{P_0, P_1, \ldots, P_q\}$ - a $(q+1)$-arc in T;

- L_i, – a line in H_i meeting T in P_i, $i = 0, 1, \ldots, q$;

- $B = \cup_{i=1}^{q}(L_i \setminus P_i)$ is a blocking set in $\Omega \setminus H_0 \cong \mathrm{AG}(n,q)$ meeting the Jamison-Bruen bound.

- Remark. In this case $t + n = q + 2$.



$\mathrm{AG}(n,q)$
H_0
L_0
$\mathrm{AG}(n-1,q)$
H_1
L_1
H_2
$\mathrm{PG}(n-2,q)$
$\mathcal{C}$
H_q
L_q

Theorem 6. For every $s = 0, 1, \ldots, q - 1 - n$, there exists an affine blocking set with parameters

$$(q^2 - s(n - 2 + s), q - (n - 2 + s))$$

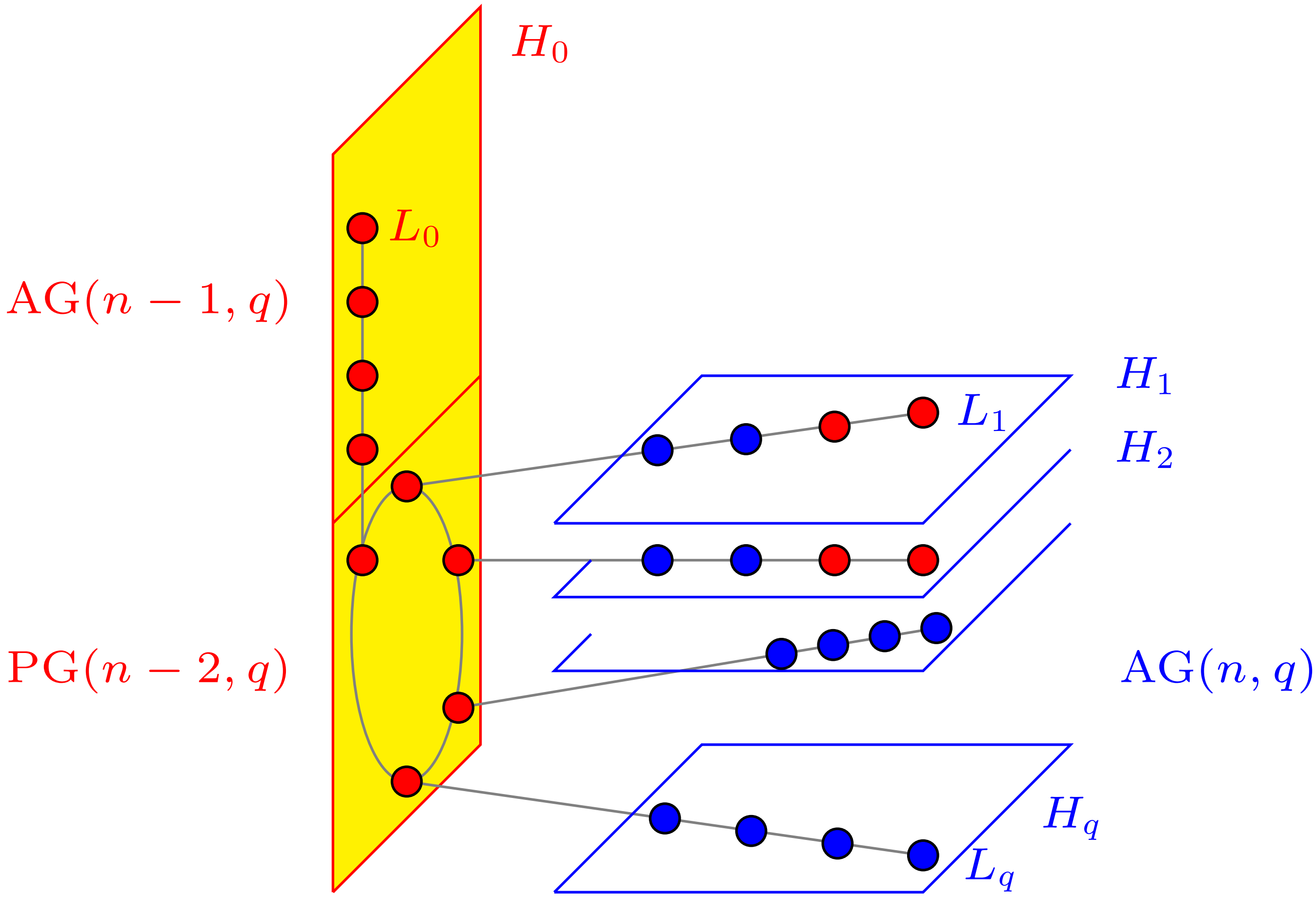
in $\mathrm{AG}(n, q)$.

Remove $n - 2 + s$ points from each of the lines $L_1, \ldots, L_s$.

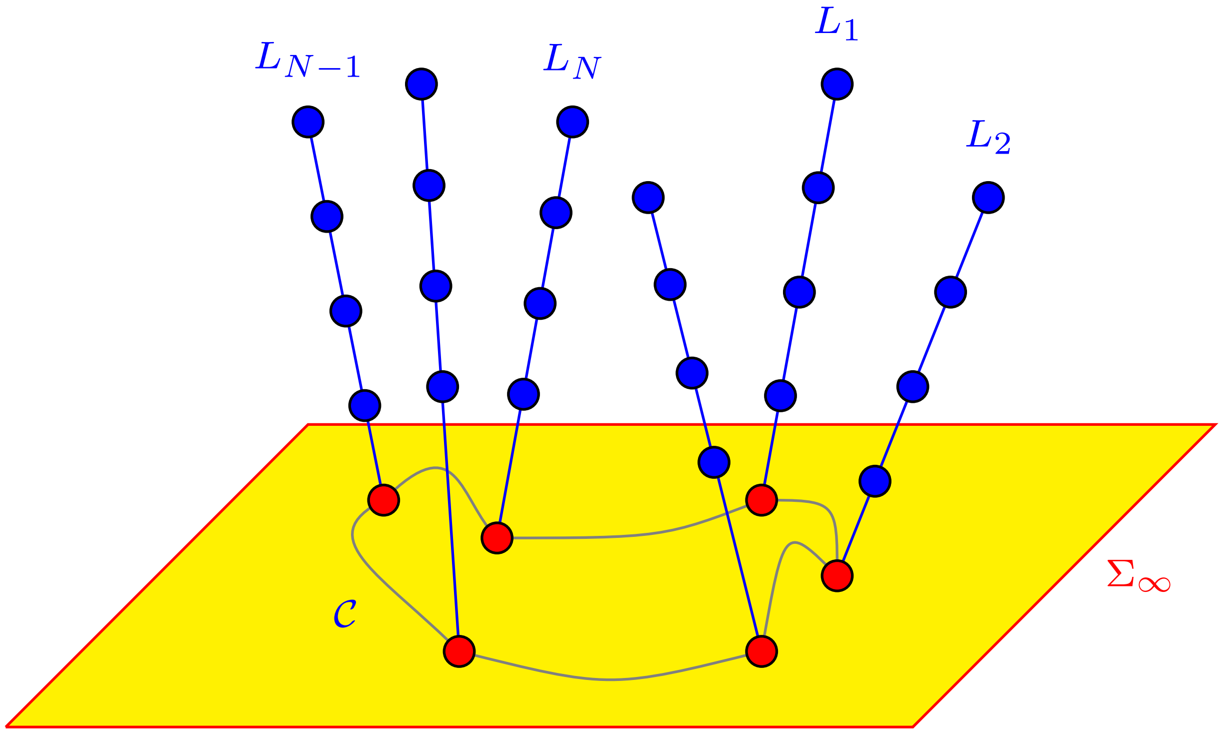
Example.

There exists a $(41, 3)$-blocking set in $\mathrm{AG}(4, 7)$.

There exists a $(45, 3)$-blocking set in $\mathrm{AG}(5, 7)$.


H_0
L_0
$\mathrm{AG}(n-1,q)$
H_1
L_1
H_2
$\mathrm{PG}(n-2,q)$
$\mathrm{AG}(n,q)$
H_q
L_q

Theorem 7. If there exists an (N, w)-arc $\mathcal{C}$ in $\mathrm{PG}(n-1, q)$ then there exists a $(qN, N-w)$-blocking set in $\mathrm{AG}(n, q)$.



4. Bounds for 3-fold blocking sets in $\mathrm{AG}(n,q)$, $n = 3, 4, 5$.

	$\mathrm{AG}(3,q)$			$\mathrm{AG}(4,q)$			$\mathrm{AG}(5,q)$		
q	LB	UB	Comment	LB	UB	Comment	LB	UB	Comment
4	16	16	Thm 2						
5	23	23	Cor 3	25	25	Thm 5			
7	33	35	Thm 7	39	41	Thm 6	45	45	Thm 6
8	36	40	Thm 7	44	48	Thm 7	52	54	Thm 6
9	41	45	Thm 7	51	54	Thm 7	57	63	Thm 7
11	53	55	Thm 7	63	66	Thm 7	73	77	Thm 7
13	63	65	Thm 7	75	78	Thm 7	87	91	Thm 7