Constructing a space-time code with a small volume

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Outline

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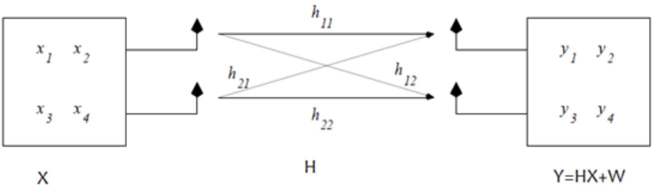
- Space-Time Block Codes (STBC)
- Multiple transmit and multiple receive antennas (MIMO)

System Model

2×2 MIMO channel



The received signal is given by

$$Y = HX + W$$

$X, H, Y, W \in M_2(\mathbb{C})$.

Code design criteria (Coherent case)

- The pairwise probability of error is bounded by

$$P(X \to \hat{X}) \leq \frac{\text{const}}{|\det(X - \hat{X})|^{2M}},$$

where M is the number of received antennas.

- We need

$$\det(X_i - X_j) \neq 0, \ \forall X_i \neq X_j, \ X_i, X_j \in \mathcal{C}$$

called fully diverse code.

The idea behind division algebras

- If $\mathcal{C}$ is taken inside an algebra of matrices, the problem simplifies to $\det(X) \neq 0, \;\; 0 \neq X \in \mathcal{C}$.
- Division algebras are rings which every nonzero element has a multiplicative inverse.

Definition (Quaternion algebra)

Let K be a field with $char\, K \neq 2$, and $a, b \in K^*$. A $K-$algebra admitting a presentation of the form

$$\langle i, j \mid i^2 = a,\ j^2 = b,\ ij = -ji \rangle$$

is called a quaternion algebra over K, we write $\left(\frac{a,b}{K}\right)$ for such an algebra.

Codes built from quaternion algebras

- Alamouti Code: $\mathcal{HA} = \left(\frac{-1,-1}{\mathbb{R}}\right)$, $i^2 = j^2 = -1$.
- Silver Code: $\mathcal{SA} = \left(\frac{-1,-1}{\mathbb{Q}(\sqrt{-7})}\right)$, $i^2 = 7$, $j^2 = -1$.
- Golden Code: $\mathcal{GA} = \left(\frac{5,i}{\mathbb{Q}(i)}\right)$, $i^2 = 5$, $j^2 = i$.

Decoding - Lattices

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Definition (Order)

Let $\mathcal{A} = \left(\frac{a,b}{K}\right)$ be a quaternion algebra and R be a ring of K. An order $\mathcal{O}$ in $\mathcal{A}$ is a subring of $\mathcal{A}$ contained 1, equivalently its a finitely generate R-module such that $\mathcal{A} = K\mathcal{O}$.

An order $\mathcal{O}$ is called maximal, if it is not properly contained in any other R-order in $\mathcal{A}$.

$$\begin{array}{ccc} \Lambda & \subset & \mathcal{A} \\ & & \mid 4 \\ & & K \\ & & \mid \\ & & \mathbb{Q} \end{array}$$

Example (Decoding- Lattice)

Let $\phi : M_2(\mathbb{C}) \to \mathbb{C}^4$ be the function that vectorizes matrices and $\{\omega_1, \omega_2, \omega_3, \omega_4\}$ a basis of an order $\mathcal{O}$ as a $\mathbb{Z}[i]$-module. Every codeword X can be written as

$$X = \sum_{i=1}^{4} s_i \omega_i, \;\; s = (s_1, s_2, s_3, s_4)^t \in \mathbb{Z}[i]^4$$

Let Φ be the matrix whose columns are $\phi(\omega_1), \phi(\omega_2), \phi(\omega_3)$, $\phi(\omega_4)$. Then the lattice point corresponding to X is

$$x = \phi(X) = \sum_{i=1}^{4} s_i \phi(\omega_i) = \Phi_s.$$

- We are interested in finding a reduced basis for the lattice generated by the channel code matrix.

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Algebraic Reduction

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Normalization of the received signal

The channel matrix H can be rewritten as

$$H = \sqrt{det(H)} H_1, \ \ H_1 \in SL_2(\mathbb{C}).$$

Therefore the received signal now is given by

$$Y_1 = \frac{Y}{\sqrt{(H)}} = H_1 X + W_1$$

Algebraic reduction: consists in approximating the matrix H_1 with a unit U of norm 1 of a maximal order $\mathcal{O}$.

- **General Case:** $H_1 = EU$, E : error

Then $E = H_1 U^{-1}$ and we require that the Frobenius norm $||E^{-1}||_F^2 = ||E||_F^2$ should be minimized:

$$\hat{U} = \underset{U \in \mathcal{O},\, det(U)=1}{\operatorname{argmin}} ||UH_1^{-1}||_F^2$$

This criterion corresponds to minimizing the trace of the covariance matrix (power) of the new noise n :

$$tr(Cov(n)) = \frac{N_0}{\det(H)}||E^{-1}||_F^2,$$

where N_0 is the variance.

Outline

Action of the group on the hyperbolic space $\mathbb{H}^3$

Poincaré's theorem establishes a correspondence between a set of generators of the group and the isometries which map a facet of the polyhedron to another facet. All the polyhedra are isometric, and they cover the whole space $\mathbb{H}^3$, forming a tiling.

- $\mathbb{H}^3 = \{(z, r) \mid z \in \mathbb{C},\ r \in \mathbb{R},\ r > 0\}$ (upper half-space model)
- $\mathbb{H}^3$ endowed with the hyperbolic distance ρ such that if

$P = (z, r),\ P' = (z', r'),$

$$\cosh \rho(P, P') = 1 + \frac{|z - z'|^2 + (r - r')^2}{2rr'}$$

Consider the action of $PSL_2(\mathbb{C}) = SL_2(\mathbb{C})/\{1, -1\}$ on the point

$$J = (0, 1)$$

which has the following property:

$$\forall g \in SL_2(\mathbb{C}),\ ||g||_F^2 = 2\cosh \rho(J, g(J)).$$

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- $g = uh_1^{-1},\ h_1 \in SL_2(\mathbb{C})$

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- Approach the points into $\mathbb{H}^3$ by the closer unit.
- Small volume $\rightarrow$ units are closer to each other.
- Volume $\rightarrow$ depends on the choice of the order.

Let $\mathcal{O}^1$ be the group of units of the maximal order $\mathcal{O}$ and $\mathcal{P}$ a compact fundamental polyhedron.

Theorem 1. (Tamagawa Volume Formula)

Let $\mathcal{A}$ be a quaternion algebra over K such that $\mathcal{A} \otimes_{\mathbb{Q}} \mathbb{R} \cong M_2(\mathbb{C})$. Let $\mathcal{O}$ be a maximal order of $\mathcal{A}$. Then the hyperbolic volume is given by,

$$Vol(\mathcal{P}_{\mathcal{O}^1}) = \frac{1}{4\pi^2}\zeta_K(2)|D_K|^{3/2} \prod_{p|\delta_{\mathcal{O}}} (N(p) - 1).$$

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Outline

Constructing a space-time code with a small volume

We propose to construct a quaternion algebra

$$\mathcal{A} = \left(\frac{2+\omega, -\omega}{\mathbb{Q}(\omega)}\right),$$

- $i^2 = 2 + \omega$
- $j^2 = -\omega$
- $\omega = (-1 + \sqrt{-3})/2$

- Maximal order:

$$\mathcal{O} = \mathbb{Z}[\omega] \oplus \mathbb{Z}[\omega]\theta \oplus \mathbb{Z}[\omega]e \oplus \mathbb{Z}[\omega]\delta$$

where $\delta = \omega + (\omega+1)\theta + (\omega+1)e + \theta e, \quad \theta = \sqrt{2+\omega}$
and $e = \begin{pmatrix} 0 & 1 \\ -\omega & 0 \end{pmatrix}$.

	Golden Code algebra	New algebra
$\zeta_K(2)$	$1.50670301\cdots$	$1.285190\cdots$
$\lvert D_K \rvert$	4	3
$\prod_{p \mid \delta_{\mathcal{O}}} (N(p) - 1)$	16	6

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$\zeta_K(2)$	$1.50670301\cdots$	$1.285190\cdots$
$\|D_K\|$	4	3
$\prod_{p\|\delta_{\mathcal{O}}}(N(p)-1)$	16	6
$Vol(P_{\mathcal{O}^1})$	$4.885149838\cdots$	$1.0338314\cdots$

- $\left|\mathcal{O}^*/\mathcal{O}^1\right| = 6$ $(\mathbb{Z}_6 \cong \{1, -1, \omega, -\omega, \omega^2, -\omega^2\} \cong \mathcal{O}^*/\mathcal{O}^1)$.
- $\#\{\text{unitary units}\} = 4$.
- The group of unitary units stabilize $J = (0, 1)$.
- Action of $PSL_2(\mathbb{C})$ on the point PJ, $P \in SL_2(\mathbb{C})$, such that the stabilizer of PJ is $\{1, -1\}$.
- $||uh_1||_F^2 = 2\cosh(\rho(\underbrace{PuP^{-1}}\ \underbrace{Ph_1^{-1}P^{-1}}PJ, PJ))$.

- $PJ = (0.00002, 1.00002)$
- faces $= 26$
- edge $= 72$
- Generators: $\{u, g_1, g_2\}$ where

A. Page, *Computing arithmetic Kleinian groups*, Submitted, on 1 Jun 2012.

$http://www.eleves.ens.fr/home/page/index-en.html$

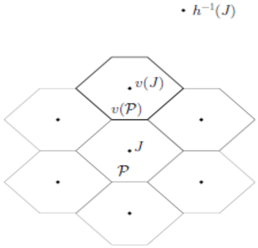
$$u = \begin{pmatrix} 0 & \omega \\ -\omega^2 & 0 \end{pmatrix}$$

$$g_1 = \begin{pmatrix} -1 - \frac{\theta}{2} - \frac{\omega}{2} - \frac{\theta\omega}{2} & -\frac{1}{2} + \frac{\theta}{2} \\ -1 - \omega + \frac{\theta\omega}{2} - \frac{\omega^2}{2} & -1 + \frac{\theta}{2} - \frac{\omega}{2} + \frac{\theta\omega}{2} \end{pmatrix}$$

$$g_2 = \begin{pmatrix} -\frac{\theta}{2} - \frac{\omega}{2} - \frac{\theta\omega}{2} & \frac{1}{2} + \frac{\theta}{2} - \omega \\ -\omega + \frac{\theta\omega}{2} + \frac{\omega^2}{2} & \frac{\theta}{2} - \frac{\omega}{2} + \frac{\theta\omega}{2} \end{pmatrix}$$

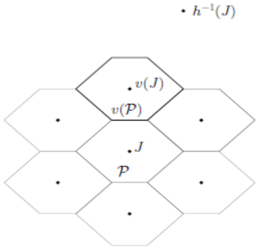
$$\omega = (1 + \sqrt{-3})/2, \;\; \theta = \sqrt{2 + \omega}$$

The Algorithm: idea



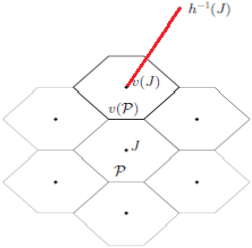
Consider $u_1, \cdots, u_r$ the generators of $\mathcal{O}^1$ and their inverses. The neighboring polyhedra of $\mathcal{P}$ are all the form $u_i(\mathcal{P})$, $i = 1, \cdots, r$.

The Algorithm: idea



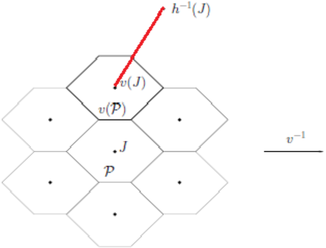
The idea is choose the u_i such that $u_i(J)$ is closest to $h_1^{-1}(J)$.

The Algorithm: idea



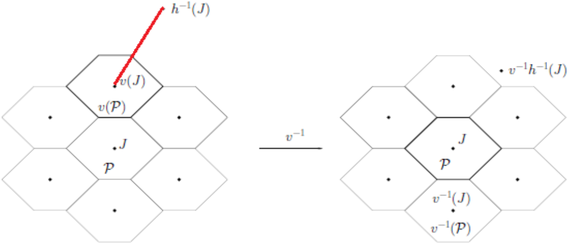
The idea is choose the u_i such that $u_i(J)$ is closest to $h_1^{-1}(J)$.

The Algorithm: idea



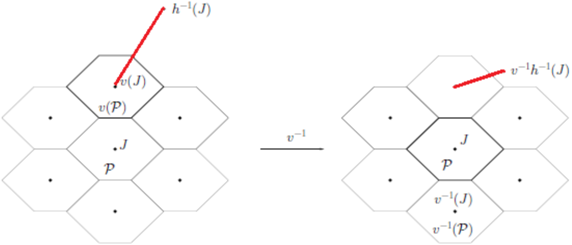
Since u_i is an isometry of $\mathbb{H}^3$, at the next step we can apply u_i^{-1}

The Algorithm: idea



Start again the search of the $u_{i'}$ that gives the closest point to $u_i^{-1} h_1^{-1}(J)$.

The Algorithm: idea



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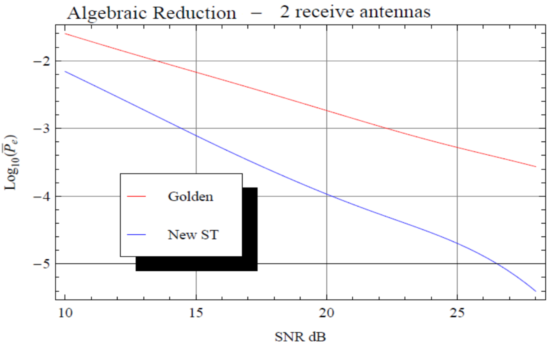
Algebraic Reduction – 2 receive antennas
$\mathrm{Log}_{10}(\overline{P}_e)$
−2
−3
−4
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25
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Outline

Further research

- Change the center J of the domain of Dirichlet and performance analysis
- Show that the new algebra introduced is an algebra space-time code with good shape.

- Explicit the fundamental domain, vertices and relations.
- Generalize the algebraic reduction to higher-dimensional space-times codes based on division algebras.

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