Sylow p-subgroups of commutative group algebras of finite abelian p-groups

Velika N. Kuneva, Todor Zh. Mollov and Nako A. Nachev

Thirteenth International Workshop on Algebraic and Combinatorial Coding Theory (ACCT)

June 15-21, 2012, Pomorie, Bulgaria

1. Introduction

Let RG be the group algebra of a finite abelian p-group G over a direct product R of finitely many commutative indecomposable rings with identities. Suppose that $V(RG)$ is the group of normalized units in RG and $S(RG)$ is the Sylow p-subgroup of $V(RG)$. In the present paper we establish the structure of $S(RG)$ when p is an invertible element in R. This investigation extends a result of Mollov [2] (1986) (Zbl 0655.16004) who gives a description, up to isomorphism, of the torsion subgroup of $V(RG)$, when R is a field of characteristic different from p. Our description is obtained owing to series of results of Mollov and Nachev. This paper is an announce of results of our article which is accepted for a publication in C. R. Acad. Bulgar. Sci. (Kuneva, Mollov and Nachev [1]).

2. Preliminary results

Let R^* be the unit group of a ring R and let p be a prime. Let $\alpha \in L$ be an algebraic element over the ring R and α be a root of a polynomial $f(x) \in R[x]$ of degree n. We say that $f(x)$ is a minimal polynomial of α over R if α is not a root of a polynomial over R which degree is less than n. We denote by $R[\alpha]$ the intersection of all subrings of L containing R and α.

2. Preliminary results

Definition

A ring R of characteristic different from the prime p is called a ring of the first kind with respect to p, if there exists a natural $j, j \geq 2$ such that $R[\varepsilon_j] \neq R[\varepsilon_{j+1}]$. In the contrary R is called a ring of the second kind with respect to p.

This definition implies immediately that if R is a ring of the second kind with respect to 2, then $R[\varepsilon_2] = R[\varepsilon_j]$ for every natural $j \geq 2$.

2. Preliminary results

Nachev ([6], Corollary 5.2) proves the following result (slightly modified).

Theorem 2.1

Let R be a commutative indecomposable ring with 1 and the prime p be invertible in R. If R is a ring of the first kind with respect to p, then there exists $i \in \mathbb{N}$, such that if $p \neq 2$, then

$$R[\varepsilon_1] = R[\varepsilon_2] = ... = R[\varepsilon_i] \neq R[\varepsilon_{i+1}] \neq ...$$

and if $p = 2$, then

$$R[\varepsilon_2] = R[\varepsilon_3] = ... = R[\varepsilon_i] \neq R[\varepsilon_{i+1}] \neq ...;$$

If R is a ring of the second kind with respect to p and $p \neq 2$, then $R[\varepsilon_1] = R[\varepsilon_j]$ for every $j \in \mathbb{N}$.

2. Preliminary results

When R is a ring of the first kind with respect to prime p, then the number i, defined in the last theorem, is called *a constant of the ring R with respect to p*.

Let η_n be a fixed root of a monic indecomposable divisor of the cyclotomic polynomial $\Phi_n(x)$ over R. We can note that if $n = p^k$, then $\eta_{p^k} = \varepsilon_k$. Suppose $G(d)$, $d \in \mathbb{N}$, is the number of the elements of order d in G and $a(d) = G(d)/[R[\varepsilon_d] : R]$, where $[R[\varepsilon_d] : R]$ is the dimension of the free module $R[\varepsilon_d]$ over the ring R.

3. Main results

If G is an abelian p-group and $k \in \mathbb{N}$, then we denote

$$G[p^k] = \left\{ g \in G \mid g^{p^k} = 1 \right\}.$$

Let $\coprod_n G$ and $\sum_n^{\cdot} R$, where $n \in \mathbb{N}$, denote the coproduct of n copies of G and the direct sum of n copies of R, respectively.

3. Main results

Theorem 3.1

Let G be a finite abelian p-group of an exponent p^n $(n \in \mathbb{N})$, R be a commutative indecomposable ring with identity, $p \in R^*$ and let R be a ring of the second kind with respect to p.

1) If either $p \neq 2$, or $p = 2$ and $R = R[\varepsilon_2]$, then

$$S(RG) \cong \coprod_{(|G|-1)/(R[\varepsilon_1]:R]} Z(p^\infty).$$

2) If $p = 2$ and $R \neq R[\varepsilon_2]$, then

$$S(RG) \cong \coprod_{|G[2]|-1} Z(2) \times \coprod_{|G \setminus G[2]|/2} Z(2^\infty).$$

For the proof we use Theorem 2.1 and the following three results.

Theorem A (Mollov and Nachev (2006)([3], Remark 4.5)). *Let G be a finite abelian group of exponent n and let R be a commutative indecomposable ring with identity. If n is an invertible element in R, then*

$$RG \cong \dot{\sum_{d/n}} a(d)R[\eta_d].$$

Theorem B (Nachev (in press) [6]. *Let R be a commutative indecomposable ring with* 1 *and the prime p be invertible in R. Then the p-component R_p of the unit group R^* is a cocyclic group.*

Theorem C (Nachev (2005) [5]). *Let R be a commutative indecomposable ring with identity and α be an algebraic element over R, such that its minimal polynomial over R is monic and indecomposable. Then $R[\alpha]$ is indecomposable.*

Theorem 3.2

Let G be a finite abelian p group of exponent p^n ($n \in \mathbb{N}$), R be a commutative indecomposable ring with identity, $p \in R^*$ and let R be a ring of the first kind with respect to p with constant i with respect to p.

1) If either $p \neq 2$, or $p = 2$ and $R = R[\varepsilon_2]$, then

$$S(RG) \cong \coprod_{\delta_i} Z(p^i) \times \coprod_{k=i+1}^{n} \coprod_{\delta_k} Z(p^k),$$

$\delta_i = (|G[p^i]) - 1)/[R[\varepsilon_i] : R]$, $\delta_k = |G[p^k] \setminus G[p^{k-1}]|/[R[\varepsilon_k] : R]$, $k = i+1, ..., n$.

...

3. Main results

Theorem 3.2

...

2) If $p = 2$ and $R \neq R[\varepsilon_2]$, then

$$S(RG) \cong \coprod_{\delta_1} Z(2) \times \coprod_{\delta_i} Z(2^i) \times \coprod_{k=i+1}^{n} \coprod_{\delta_k} Z(2^k),$$

$\delta_1 = |G[2]| - 1$, $\delta_i = |G[2^i] \setminus G[2]|/[R[\varepsilon_i] : R]$, $\delta_k = |G[2^k] \setminus G[2^{k-1}]|/[R[\varepsilon_k] : R]$, $k = i+1, ..., n$.
The proof is obtained as the proof of Theorem 3.1.

Theorem 3.3

Let G be a finite abelian p-group and let $R = \prod_{i \in \mathbb{I}} R_i$, where R_i are commutative indecomposable rings with identities such that p is an invertible element in R. Then

$$S(RG) \cong (\prod_{i \in \mathbb{I}} S(R_i G))_p.$$

In particular if $R = \prod_{i=1}^{n} R_i$, then

$$S(RG) \cong \prod_{i=1}^{n} S(R_i G).$$

The description of $S(R_i G)$ is given by Theorems 3.1 and 3.2.

The proof is directly obtained by the following preposition.

Preposition (Mollov and Nachev (in press) [4]). *If G is a finite abelian group and R_i, $i \in \mathbb{I}$, are commutative rings with identities, then*

$$(\prod_{i \in \mathbb{I}} R_i)G \cong \prod_{i \in \mathbb{I}} R_i G.$$

References

[1] Kuneva V. N., Mollov T. Zh. and Nachev N. A. *Sylow p-subgroups of commutative group algebras of finite abelian p-groups*, C.R. Acad.Bulgar.Sci. (In press).

[2] Mollov T. Zh. *Multiplicative groups of semisimple group algebras*, (Russian), Pliska Stud. Math. Bulgar. **8**, 1986, 54–64.

[3] Mollov T. Zh. and Nachev N. A. *Unit groups of commutative group rings*, Communications in Algebra, **34**, 2006, 3835-3857.

[4] Mollov T. Zh. and Nachev N. A. *Isomorphism of commutative group algebras of finite abelian groups*, Plovdiv University "Paissii Hilendarsky", Bulgarian Scientific works.**38** Mathematics (In press).
[5] Nachev N. A. *Nilpotent elements and idempotents in commutative rings*, Communications in Algebra, **33**, 2005, 3631-3637.
[6] Nachev N. A., *Torsion elements in commutative indecomposable rings*, C.R. Acad.Bulgar.Sci. (In press).

Department of Algebra and Geometry,
University of Plovdiv
4000 Plovdiv, Bulgaria
e-mail: kuneva.1977@abv.bg
mollov@uni-plovdiv.bg
nachev@uni-plovdiv.bg

Thank you for the attention!