Proper integers for search with a lie

Search problem

1. Set A

2. An unknown element $x \in A$

3. A collection $\mathcal{B}$ of subsets of A which is called question set.

4. Questions of the type: Is x in $B \in \mathcal{B}$?

Adaptive and nonadaptive search

Without a lie or with at most one lie

Let $A = \{1, 2, 3, \ldots, 2^k\}$

Let S be positive integer. Question set $\mathcal{B}$ consists of all subsets of A such that the sum of elements of B equals S.

We allow at most one lie in the answers received.

S is proper if one can find x by minimum possible questions using nonadaptive search.

A vector $v_1, v_2, \ldots, v_{2^k}$ is a characteristic vector for a subset B of A if $v_i = 1$ when $i \in B$ and $v_i = 0$ otherwise.

An $n \times 2^k$ matrix G is called characteristic matrix for a collection $B_1, B_2, \ldots, B_n$ if the rows of G are all characteristic vectors of $B_1, B_2, \ldots, B_n$.

If x can be found by the question set $B_1, B_2, \ldots, B_n$ then the columns of the corresponding characteristic matrix form a binary one error-correcting code.

It is easy to see that if S is a proper integer then

$$G(1\ 2\ 3 \ldots 2^k) = S(11 \ldots 1)^t.$$

Consider one error-correcting cyclic code of length n having 2^k codewords. All codewords split into orbit matrices with respect to cyclic shift. Also if $C_{w,l}$ is an orbit matrix of length l and weigh w (each column is of weight w) then the matrix $\overline{C_{w,l}}$ is also an orbit matrix.

Lemma 1. Let $C_1, C_2, \dots C_m$ be all orbit matrices such that for any $i, 1 \leq i \leq m$, $\overline{C_i}$ is also from this collection. Then the matrix $G = C_1 C_2 \dots C_m$ is proper one.

Sketch proof. We have to show that $C_{w,l}$ and $\overline{C_{w,l}}$ add one and the same amount in the scalar product of every row of G with $(1, 2, \dots, 2^k)$

It follows that the maximum value S_{max} of proper integer S is obtained when orbit matrices are in increasing order of their weights.

S_{min} is obtained when orbit matrices are in decreasing order of their weights.

Are all integers in the interval $[S_{min}, S_{max}]$ proper?

For the binary Hamming code of length $n = 2^t - 1$ for t even, all integers in the interval $[S_{min}, S_{max}]$ are proper once.

Consider proper matrix G and let $C_{p,t}$ and $C_{q,h}$ be neighboring orbit matrices in G. Then $G_1 = G(C_{p,t}C_{q,h} \to C_{q,h}C_{p,t})$ is a proper matrix of weight $wt(G) = wt(G) + \frac{th(p-q)}{n}$.