Spherically punctured biorthogonal codes

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ACCT 2012

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Outline

1. Motivation and summary of results
2. Reed-Muller (RM) codes – review
3. Spherically punctured Hadamard codes
4. Precoding
5. Construction of good codes
6. Punctured biorthogonal codes on the sphere of radius b
7. Open problems

Motivation

- Polar codes can achieve channel capacity on very long blocks

- Consider a new class of codes
  - that is shorter
  - that keeps the polarization property – by using cancellation (recursive) decoding

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1. Consider a new class of punctured RM(r,m) codes with positions restricted to the points of the hypercube $\mathbb{F}_2^m$ that have some fixed Hamming weight
2. Codeword weight is determined by the weight of its information block. This dependence is based on the values of Krawtchouk polynomials and is rather nontrivial. Typically, the larger the input weight, the larger the output weight
3. Find parameters of punctured codes and show that the minimum weight codewords are obtained on the input weights 1 or 2
4. Precode information blocks in some simple code. This increases the weight of the input block and the obtained codeword at the expense of the code rate. Some codes attain the Griesmer bound
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Reed-Muller [Reed'54, Muller'54] and Spherically Punctured Codes

Reed-Muller (RM) Codes $\mathcal{R}(r, m)$

- Polynomial structure:

  Messages: polynomials of degree at most r in m boolean variables
  Encoding: truth table

- Parameters:

  Length $n = 2^m$. Dimension $k = \sum_{i=0}^{r} \binom{m}{i}$. Minimum distance $d = 2^{m-r}$.

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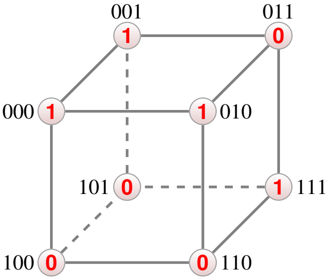
Example: $m = 3$, $r = 2$

Message $f(x_1, x_2, x_3) = x_2x_3 + x_1 + 1$

Reed-Muller code $\mathcal{R}(2,3)$



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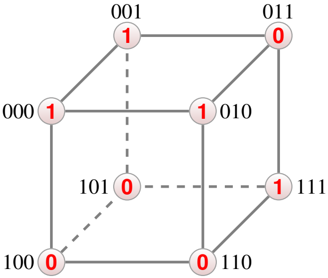

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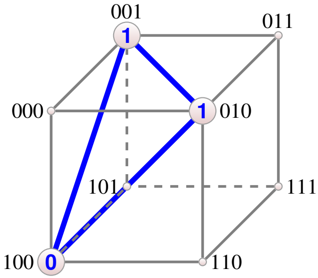

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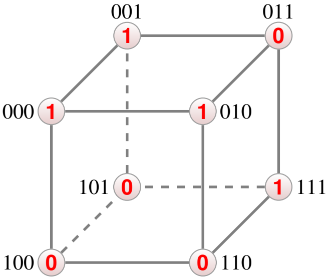

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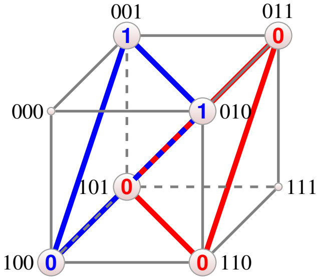

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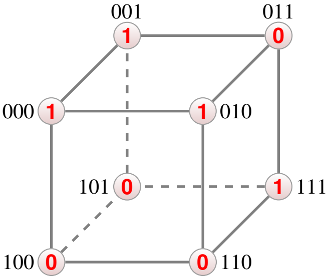

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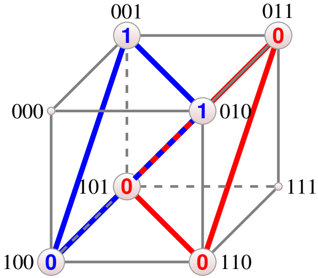

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We now consider first order punctured codes $P(1,m,b)$

Spherically punctured Hadamard codes $H(m,b)$

The Hadamard codes $H(m)$:

- Formed by all linear functions of m variables

$$f(x_1,\ldots,x_m)=\sum_{i=1}^{m} f_i x_i \qquad f_i,x_i\in\{0,1\}$$

- Parameters: $n=2^m \quad k=m \quad d=2^{m-1}$

Definition

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The Hadamard code $H(4)$:

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$n=\binom{4}{2}=6$

The Punctured Hadamard code $H(m,2)$

Consider code $H(m,2)$. Ease to see that

- $H(m,2)$ has length $\binom{m}{2}$
- Codeword weight is determined by message weight

Lemma

Let $f(x) = \sum_{i=1}^{m} f_i x_i$ be a message such that wt$(f_1, f_2, \ldots, f_m) = w$.
Then codeword weight is $w(m-w)$

Pick x : wt$(x) = 2$

$$\begin{matrix} x_i \rightarrow \\ \\ x_j \rightarrow \end{matrix} \begin{bmatrix} 1 \\ \\ 1 \end{bmatrix}$$

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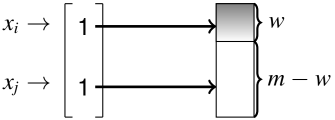

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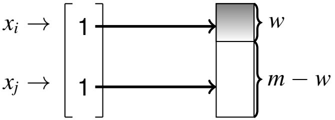

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Corollary

- *Minimum distance of* $H(m,2)$ *is* $m-1$ *and is achieved at* $w = 1$
- *Dimension of* $H(m,2)$ *is* $m-1$ *(vector with* $w = m$ *gives zero codeword)*

Precoding

Motivating example:

- Consider code $H(m,\{1,2\})$ on spheres of radii $b=1,2$
- Message weight w
- Codeword weight $w + w(m-w)$

$$G_{H(4,\{1,2\})} = \left[\begin{array}{cccc|cccccc} 1&0&0&0&0&0&0&1&1&1\\ 0&1&0&0&0&1&1&0&0&1\\ 0&0&1&0&1&0&1&0&1&0\\ 0&0&0&1&1&1&0&1&0&0 \end{array}\right]$$

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For example:

- if input alphabet is $\mathbb{F}_2^m$, then $d(m,\{1,2\}) = m$
- if input alphabet is parity check code $G[m,m-1,2]$, then $d(m,\{1,2\}) = 2m-2$

General scheme:

$$u \in \mathbb{F}_2^k \xrightarrow{\text{Precoding}} g \in \mathbf{G} \xrightarrow{H(m,B)} y \in H_{\mathbf{G}}(m,B)$$

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Precoding

Motivating example:

- Consider code $H(m, \{1,2\})$ on spheres of radii $b = 1, 2$
- Message weight w
- Codeword weight $w + w(m-w)$

$$G_{H(4,\{1,2\})} = \underbrace{\left[\begin{array}{cccc|cccccc} 1&0&0&0&0&0&0&1&1&1 \\ 0&1&0&0&0&1&1&0&0&1 \\ 0&0&1&0&1&0&1&0&1&0 \\ 0&0&0&1&1&1&0&1&0&0 \end{array}\right]}_{b=1 \quad\quad b=2}$$

The more concentrated the input spectrum, the higher the minimum distance $d(m, \{1,2\})$

For example:

- if input alphabet is $\mathbb{F}_2^m$, then $d(m, \{1,2\}) = m$
- if input alphabet is parity check code $G[m, m-1, 2]$, then $d(m, \{1,2\}) = 2m - 2$

General scheme:

$$u \in \mathbb{F}_2^k \xrightarrow{\text{Precoding}} g \in \mathbf{G} \xrightarrow{H(m,B)} y \in H_{\mathbf{G}}(m, B)$$

Precoding allows to build codes that attain Griesmer bound

Griesmer bound: for linear $[n,k,d]$ binary code $n \geq \sum_{i=0}^{k-1} \left\lceil \frac{d}{2^i} \right\rceil$

Lemma

Let $G(s) = [2^s - 1, s, 2^{s-1}]$ be the shortened RM$(1,s)$ code. Then $H_{G(s)}(2^s - 1, \{1,2\})$ meets the Griesmer bound

- $H_{G(s)}(2^s - 1, \{1,2\})$ has dimension s
- $H_{G(s)}(2^s - 1, \{1,2\})$ has length $2^{2s-1} - 2^{s-1}$
- Each precoded message has weight 2^{s-1}
- Each codeword has weight 2^{2s-2}

Also, $H_{G(s)}(2^s - 1, B)$ for $B = \{1, 2, 2^s - 2, 2^s - 3\}$ (or any its subset) attains the Griesmer bound

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Spherically punctured biorthogonal codes for general b

- Recall that first order RM$(1, m)$ code is formed by all affine functions of m variables

$$f(x_1, \ldots, x_m) = f_0 + \sum_{i=1}^{m} f_i x_i \qquad f_i, x_i \in \{0, 1\}$$

and has parameters : $n = 2^m \quad k = m + 1 \quad d = 2^{m-1}$

- Thus

$$\mathcal{R}(1, m) = H(m) \cup \{H(m) + \mathbf{1}\}$$

Definition

Spherically punctured biorthogonal code $P(m, b)$ is the code $\mathcal{R}(1, m)$ punctured to positions x so that wt$(x) = b$

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Easy to see that:

- $P(m, b)$ has length $\binom{m}{b}$
- codeword weight is determined by message weight

Consider a message $f(x)$ whose linear part has weight w:

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Lemma

Codeword weight is

$$\frac{1}{2}\left(\binom{m}{b} - (-1)^{f_0} K_b^m(w)\right),$$

where $K_b^m(w)$ is the binary Krawtchouk polynomial defined as

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We now study the minimum distance $d(m, b)$ of $P(m, b)$

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Minimum distance of $P(m, b)$

Theorem

Spherically punctured biorthogonal code $P(m, b)$ has

- *length* $\binom{m}{b}$
- *dimension* m
- *minimum distance*

$$d(m,b) = \begin{cases} \binom{m-1}{b-1}, & \text{if } m > 2b \\ \binom{m-1}{b}, & \text{if } m < 2b \\ 2\binom{m-2}{b}, & \text{if } m = 2b \end{cases}$$

- Finding the minimum distance of $P(m, b)$ is much more involved that for standard RM codes
- In RM(r, m) analysis, a hypercube is split into two identical subcubes, that behave similarly and yield a recursive estimate $(u \| u + v)$
- In our case, a sphere decomposes into two different subspheres. Furthermore, the subspheres may behave differently in terms of minimum distance

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Minimum distance of $P(m, b)$

Split the sphere $S(m, b) = \{(x_1, \ldots, x_m) : \text{wt}(x) = b\}$ into two sub-spheres

- $S(m-1, b)$ that consists of $(x_1, \ldots, x_m) \in S(m, b)$ that have $x_m = 0$
- $S(m-1, b-1) = S(m, b) - S(m-1, b)$

Thus, $P(m, b)$ decomposes as:

$$(m-1, b-1) \nwarrow (m, b) \nearrow (m-1, b)$$

Node (m, b) might decompose into

- nodes of same types – easy case
- nodes of different types – hard case
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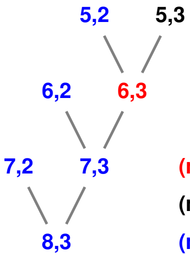


(m,b): $m = 2b$

(m,b): $m < 2b$

(m,b): $m > 2b$

Minimum of Krawtchouk polynomials

Corollary

For $b \in [1, m-1]$, and $w \in [1, m-1]$

$$\max\{|K_b^m(1)|, |K_b^m(2)|\} \geq |K_b^m(w)|$$

Similar result was previously known only in asymptotic setting for large m and linearly growing b

We now study ways to improve $d(m, b)$

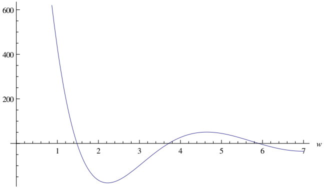
Precoding in the general case

Krawtchouk polynomial $K_b^m(w)$

- has b simple roots $r_1 < r_2 < \ldots < r_b$ so that

$$r_1 \geq \frac{m}{2} - \sqrt{b(m-b)}$$

- decays in $[0, r_1]$ and oscillates in $[r_1, r_b]$
- is 'small' in the oscillating region, i.e. $|K_b^m(w)| \leq 2^{-m\theta/2}\binom{m}{b}, \ \theta > 0$



Good precoding would concentrate the weight spectrum close to the oscillating region

Thus, if the input spectrum of G is contained within $[\delta_{\min}, \delta_{\max}]$, then

$$d(m,b) \geq \frac{1}{2}\left(\binom{m}{b} - \max\left\{K_b^m(\delta), 2^{-m\theta/2}\binom{m}{b}\right\}\right), \delta = \begin{cases} \delta_{\min}, & \text{odd } b \\ \min\{\delta_{\min}, m - \delta_{\max}\}, & \text{even } b \end{cases}$$

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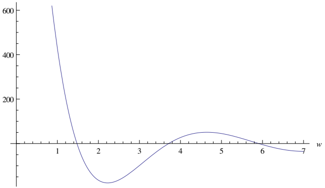

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