Formally Self-Dual Codes and Gray Maps

Steven T. Dougherty

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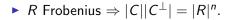
• Linear code of length n – submodule of R^n

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$$\mathbf{v}, \mathbf{w} = \sum \mathbf{v}_i \overline{\mathbf{w}_i}$$
$$\mathbf{c}^{\perp} = \{ \mathbf{v} \mid [\mathbf{v}, \mathbf{w}] = \mathbf{0}, \forall \mathbf{w} \in C \}$$



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• $W_C(y) = W_{C^{\perp}}(y)$ the code is formally self-dual.

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Rings

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$$\mathbb{Z}_4 = \{0, 1, 2, 3\}$$

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• $R_k = \mathbb{F}_2[u_1, u_2, \dots, u_k] / \langle u_i^2 = 0, u_i u_j = u_j u_i \rangle$

$$\phi_{\mathbb{Z}_4}: \mathbb{Z}_4 \to \mathbb{F}_2^2$$

$$egin{array}{rcl} \phi_{\mathbb{Z}_4}(0) &=& (00) \ \phi_{\mathbb{Z}_4}(1) &=& (01) \ \phi_{\mathbb{Z}_4}(2) &=& (11) \ \phi_{\mathbb{Z}_4}(3) &=& (10) \end{array}$$

$$\phi_{R_1}(a + bu_1) = (b, a + b)$$

 $\phi_{R_k}(a + bu_k) = (\phi_{R_{k-1}}(b), \phi_{R_{k-1}}(a) + \phi_{R_{k-1}}(b))$

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The maps ϕ_{R_k} and ϕ_{A_k} are linear but the map $\phi_{\mathbb{Z}_4}$ is not. The Lee weight is the Hamming weight of its binary image.

Inner Products

Over A_k , the Euclidean inner product is:

$$[\mathbf{v},\mathbf{w}] = \sum \mathbf{v}_i \mathbf{w}_i$$

and the Hermitian is

$$[\mathbf{v},\mathbf{w}]_H = \sum \mathbf{v}_i \overline{\mathbf{w}_i}$$

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where $\overline{v_i} = 1 + v_i$.

Theorem

If C is a formally self-dual code over \mathbb{Z}_4 , R_k or A_k then the image under the corresponding Gray map is a binary formally self-dual code.

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Major Result

Theorem

Let C be an odd formally self-dual binary code of even length n. Let C_0 be the subcode of even vectors. The code $C = \langle \{(0,0,\mathbf{c}) \mid \mathbf{c} \in C_0\} \cup \{(1,0,\mathbf{c}) \mid \mathbf{c} \in C - C_0\}, (1,1,1) \rangle$ is an even formally self-dual code of length n + 2 with weight enumerator $W_{\overline{C}} = x^2 W_{C_{0,0}}(x, y) + xy W_{C_{1,0}}(x, y) + y^2 W_{C_{0,0}}(y, x) + xy W_{C_{1,0}}(y, x).$ The code $\overline{C} = \langle \{(0,0,\mathbf{c}) \mid \mathbf{c} \in C_0\} \cup \{(1,1,\mathbf{c}) \mid \mathbf{c} \in C - C_0\}, (1,0,\mathbf{1}) \rangle$ is an odd formally self-dual code of length n + 2 with weight enumerator: $W_{\overline{C}} = x^2 W_{C_{0,0}}(x, y) + y^2 W_{C_{1,0}}(x, y) + xy W_{C_{0,0}}(y, x) + xy W_{C_{1,0}}(y, x).$ Moreover, any code with these weight enumerators is a formally self-dual code.

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- ▶ Let *C* be an odd formally self-dual code.
- ► There exists a vector t such that C = (C₀, t), where C₀ is the subcode of even vectors.

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$$\overline{C}$$
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Odd formally self-dual codes

There exist odd formally self-dual codes of all lengths over A_k for all k.

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Odd formally self-dual codes

- There exist odd formally self-dual codes of all lengths over A_k for all k.
- ► Linear odd formally self-dual codes exist over Z₄ and R_k for all lengths greater than 1.

Let **2** be the all 2 vector in \mathbb{Z}_4^n , $\mathbf{u_1}\mathbf{u_2}...\mathbf{u_k}$ be the all $u_1u_2...u_k$ vector in \mathbb{R}_k^n and **1** be the all one-vector (over any ring). Note that the Gray image of these vectors is the binary all-one vector.

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Theorem

Let C be a formally self-dual code. The code C is even over \mathbb{Z}_4 if and only if $\mathbf{2} \in C$. The code C is even over R_k if and only if $\mathbf{u}_1\mathbf{u}_2\ldots\mathbf{u}_k \in C$. The code C is even over A_k if and only if $\mathbf{1} \in C$.

Formally self-dual codes

Theorem

Let C be an odd formally self-dual code over A_k or \mathbb{Z}_4 of length n. Then C is a neighbor of an even formally self-dual code.

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Importance of these codes

 Formally self-dual codes over R_k produce binary formally self-dual codes that have k distinct automorphisms

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Importance of these codes

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- ► Formally self-dual codes over Z₄ produce non-linear formally self-dual codes which may have higher minimum distance than any linear formally self-dual codes.

Importance of these codes

- Formally self-dual codes over R_k produce binary formally self-dual codes that have k distinct automorphisms
- ► Formally self-dual codes over Z₄ produce non-linear formally self-dual codes which may have higher minimum distance than any linear formally self-dual codes.
- ► A formally self-dual code over A_k can be constructed using any 2^{k-1} binary codes.