Thirteenth International Workshop on Algebraic and Combinatorial Coding Theory

An improved algorithm for proving nonexistence of small spherical designs

Peter Boyvalenkov, Maya Stoyanova

Pomorie, Bulgaria, 15-21 June, 2012

Spherical designs

P. Delsarte, J.-M. Goethals, J. J. Seidel, Spherical codes and designs, Geom. Dedicata 6, 1977, 363-388.

Definition 1. A finite nonempty subset $C \subset \mathbb{S}^{n-1}$ is called a spherical τ-design if for every point $x \in C$ and for every real polynomial $f(t)$ of degree at most τ, the equality

$$\sum_{y \in C \setminus \{x\}} f(\langle x, y \rangle) = f_0 |C| - f(1) \qquad (1)$$

holds, where f_0 is the first coefficient in the expansion of $f(t) = \sum_{i=0}^{k} f_i P_i^{(n)}(t)$ in terms of the Gegenbauer polynomials. The strength of C is the maximal number $\tau = \tau(C)$ such that C is a spherical τ-design.

We have the following formula

$$f_0 = a_0 + \sum_{i=1}^{[k/2]} \frac{a_{2i}(2i-1)!!}{n(n+2)\cdots(n+2i-2)} = a_0 + \frac{a_2}{n} + \frac{3a_4}{n(n+2)} + \cdots .$$

Problems

Problem 1. For fixed strength $\tau \geq 3$ and dimension $n \geq 3$ find bounds on the quantity

$$B_{\text{odd}}(n,\tau) = \min\{M = |C| : |C| \text{ is odd and } \exists\ \tau\text{-design } C \subset \mathbb{S}^{n-1}\}.$$

Problem 2. Decide whether a τ-design on $\mathbb{S}^{n-1}$ of odd cardinality $M = |C|$ exists for fixed strength τ, dimension n and M.

Problem 3. (asymptotic) For a fixed odd integer $\tau = 2k-1 \geq 3$ and for $n \to \infty$ obtain lower bounds on $B_{\text{odd}}(n,\tau)$,
More precisely, we want to find new A_τ where
$B_{\text{odd}}(n,\tau) \gtrsim A_\tau \dfrac{2n^{k-1}}{(k-1)!}$, A_τ depending on τ only, if
$\liminf_{n\to\infty} \dfrac{(k-1)!B_{\text{odd}}(n,\tau)}{2n^{k-1}} \geq A_\tau$.

Some known results

Delsarte-Goethals-Seidel bounds

P. Delsarte, J.-M. Goethals, J. J. Seidel, Spherical codes and designs, Geom. Dedicata 6, 1977, 363-388.

$$B(n,\tau) \geq D(n,\tau) = \begin{cases} 2\binom{n+k-2}{n-1}, & \text{if } \tau = 2k-1, \\ \binom{n+k-1}{n-1} + \binom{n+k-2}{n-1}, & \text{if } \tau = 2k. \end{cases}$$

$$B_{\mathsf{odd}}(n, 2k-1) \gtrsim \frac{2n^{k-1}}{(k-1)!}.$$

Some known results

B. Reznick, 1995, *Lin. Alg. Appl.*
Constructions of spherical 5-designs in three dimensions for cardinalities 16, 18, 20, 22, 24 and ≥ 26.

R. H. Hardin, N. J. A. Sloane, 1996, *Discr. Comp. Geom.*
Constructions of some spherical designs in three dimensions.

B. Bajnok, 1998, *Graphs Combin.*, 2000, *Des. Codes Crypt.*
Constructions of 3-designs on $\mathbb{S}^{n-1}$ with all admissible even cardinalities (i.e. $\geq 2n$) and all odd cardinalities $M \geq 5n/2$ for $n \geq 6$, and to 11 for $n = 3, 4$, and 15 for $n = 5$.

Some known results

P. Boyvalenkov, S. Nikova, 1993-1994, *Springer-Verlag Lect. Notes Comp. Science.*

Results - new lower bounds on $B(n, \tau)$ for $\tau \geq 6$.

V. Yudin, 1997, *Izv.: Math.*

Results - some lower bounds on $B(n, \tau)$.

G. Fazekas, V. I. Levenshtein, 1997, *J. Combin. Theory.*

Restrictions on the structure of spherical designs.

P. Boyvalenkov, D. Danev, S. Nikova, 1998, *Discr. Comput. Geom.*

Nonexistence results of spherical designs with odd strength τ and odd cardinality $|C|$.

Complete solution of Problem 2 for $\tau = 3$ in dimensions $n = 4$ and $n = 6$.

Some known results

P. Boyvalenkov, S. Boumova, D. Danev, 1999, *Europ. J. Combin.*

Necessary Condition: *If $C \subset \mathbb{S}^{n-1}$ is a τ-design with odd $\tau = 2e-1$ and odd $|C|$ then $\rho_0|C| \geq 2$.*

P. Boyvalenkov, S. Boumova, D. Danev, 2002, *Proc. CTF.*

Nonexistence in 50 cases of spherical 3-designs with odd $|C|$, such that $\rho_0|C| \geq 2$ and $3 \leq n \leq 50$, and nonexistence in 53 cases of spherical 5-designs with odd $|C|$, such that $\rho_0|C| \geq 2$ and $3 \leq n \leq 20$, where the existence/nonexistence was not resolved.

Complete solution of Problem 2 for $\tau = 3$ in dimensions $n = 9$ and $n = 10$.

Some known results

S. Boumova, P. Boyvalenkov, H. Kulina, M. Stoyanova, 2009, *Des. Codes Crypt.*

Nonexistence in 35 (out of all possible 47) cases of spherical 3-designs with odd $|C|$, such that $2 \leq \rho_0|C| < 3$, $2\alpha_0^2 - 1 > \alpha_1$ and $3 \leq n \leq 50$, where the existence/nonexistence was not resolved.

Complete solution of Problem 2 for $\tau = 3$ in dimensions $n = 8, 13, 14$ and 18.

Nonexistence in 42 (out of all possible 42) cases of spherical 5-designs with odd $|C|$, such that $2 \leq \rho_0|C| < 3$, $2\alpha_0^2 - 1 > \alpha_2$ and $5 \leq n \leq 25$, where the existence/nonexistence was not resolved.

Some known results

S. Boumova, P. Boyvalenkov, M. Stoyanova, 2009, *Pr. Inform. Trans.*

Nonexistence in 290 (out of all possible 291) cases of spherical 7-designs with odd $|C|$, such that $2 \leq \rho_0|C| < 3$, $2\alpha_0^2 - 1 > \alpha_3$ and $3 \leq n \leq 20$, where the existence/nonexistence was not resolved. The exception – the case $n = 4$, $|C| = 43$.

P. Boyvalenkov, M. Stoyanova, 2010, *Discr. Math.*

$$B_{\mathrm{odd}}(n,\tau) \gtrsim \frac{(1+\sqrt[2k-1]{3})}{2}\frac{2n^{k-1}}{(k-1)!},$$

i.e. $A_\tau = \dfrac{(1+\sqrt[2k-1]{3})}{2}$, for $\tau = 2k-1$, $k = 3, 4, \ldots, 13$.

Our techniques

Let the integers $n \geq 3$, odd $\tau = 2k-1 \geq 3$, and odd M be fixed and let $C \in \mathbb{S}^{n-1}$ be a spherical τ-design of odd size $|C| = M$.

(Levenshtein) Then there exist uniquely determined real numbers $-1 \leq \alpha_0 < \alpha_1 < \cdots < \alpha_{k-1} < 1$ and positive $\rho_0, \rho_1, \ldots, \rho_{k-1}$ such that the equality

$$f_0 = \frac{f(1)}{M} + \sum_{i=0}^{k-1} \rho_i f(\alpha_i) \tag{2}$$

holds for every real polynomial $f(t)$ of degree at most $\tau = 2k-1$.

We denote $g(t) = \prod_{i=1}^{k-1}(t-\alpha_i)^2 = \sum_{i=0}^{2k-2} g_i P_i^{(n)}(t)$. Then (2) implies that $g_0 = \rho_0 |C| g(\alpha_0)$. We set $1 + \gamma(k-1)! := \theta$ for short.

Our techniques

We associate every point $x \in C$ with an ordered $(|C|-1)$-tuple $I(x)$ of the inner products $\langle x, y\rangle$, $y \in C \setminus \{x\}$, so that $I(x) = (t_1(x), t_2(x), \ldots, t_{|C|-1}(x))$, where $-1 \le t_1(x) \le t_2(x) \le \cdots \le t_{|C|-1}(x) < 1$.

Then equation (1) gives

$$\sum_{i=1}^{|C|-1} f(t_i(x)) = f_0|C| - f(1), \qquad (3)$$

which holds for every point $x \in C$ and for every real polynomial $f(t)$ of degree at most τ.

We denote by $U_i(x)$ (respectively $L_i(x)$) any upper (resp. lower) bound on the inner product $t_i(x)$, omitting x if the corresponding bound is valid for all $x \in C$.

Our techniques

Theorem 2. Let $C \subset \mathbb{S}^{n-1}$ be a τ-design with odd strength $\tau = 2k-1$ and odd size $M = |C|$. Then $\rho_0|C| \geq 2$ and:

a) $t_1(x) \leq U_1 = \alpha_0$ and $t_{|C|-1}(x) \geq L_{|C|-1} = \alpha_{k-1}$ hold for every point $x \in C$;

b) there exist three distinct points $x, y, z \in C$ such that $t_1(x) = t_1(y)$, $t_2(x) = t_1(z)$ and $\langle y, z\rangle \geq 2\alpha_0^2 - 1$;

c) if $M = \left(\frac{2}{(k-1)!} + \gamma\right) n^{k-1}$, where $\gamma > 0$ is a constant and n tends to infinity, then $\alpha_i \sim 0$, for $i = 1, 2, \dots, k-1$, $\theta\alpha_0 \sim -1$; $g(t) \sim t^{2k-2}$, $\rho_0|C| \sim \theta^{2k-1}$ and $\rho_0|C|g(\alpha_0) \sim \theta$.

We suppose that $2 \leq \rho_0|C| \leq 3$ (resp. $\gamma < \frac{\sqrt[2k-1]{3}-1}{(k-1)!}$) for $\tau = 3$ and $\rho_0|C| \leq 4$ (resp. $\gamma < \frac{\sqrt[2k-1]{4}-1}{(k-1)!}$) for $\tau > 3$.

The improved algorithm

Let $\{x, y, z\} \subset C$ be a special triple as in Theorem 2.b).

Note that the last inequality of Theorem 2.b) implies the bounds

$$L_{|C|-1}(y) = L_{|C|-1}(z) = \max\{2\alpha_0^2 - 1, \alpha_{k-1}\},$$

and the corresponding bound

$$L_{|C|-1}(y) = L_{|C|-1}(z) \sim \frac{2 - \theta^2}{\theta^2}$$

in the asymptotic.

We obtain consecutively the following general bounds for the above special triple $\{x, y, z\} \subset C$.

The improved algorithm

General bounds for the special triple $\{x, y, z\} \subset C$

(1) $t_1(z) \geq L_1(z)$, (with $g(t)$);

(2a) $t_2(z) \leq U_2(z)$, (with $f(t) = (t - t_2(z))g(t)$);

(2b) $t_2(y) \leq U_2(y)$, (with $f(t) = (t - t_2(y))g(t)$);

(3a) $L_3(z) \leq t_3(z)$, (with $g(t)$);

(3b) $L_3(y) \leq t_3(y)$, (with $g(t)$);

(4a) $L_4(z) \leq t_4(z)$, (with $g(t)$);

(4b) $L_4(y) \leq t_4(y)$, (with $g(t)$);

(5) $t_{|C|-1}(x) \geq L_{|C|-1}(x)$;

(6a) $t_{|C|-1}(z) \geq L_{|C|-1}(z)$;

(6b) $t_{|C|-1}(y) \geq L_{|C|-1}(y)$.

The improved algorithm

Definition 3. A special triple $\{x, y, z\}$ is called bad (in particular y-bad or z-bad) if $t_2(y) > \alpha_0$ or $t_2(z) > \alpha_0$.

We consider two cases:

Case 1. There exists at least one bad special triple.

Case 2. There are no bad triples.

For specific parameters, we need the nonexistence criteria fulfilled in both cases.

Case 1.1: There exists a z-bad special triple.

Let $\{x, y, z\} \subset C$ be a z-bad special triple, i.e. $\alpha_0 < t_2(z)$.
We obtain consecutively:

(1 zbc) $\quad t_1(z) \leq U_1(z) < \alpha_0, \quad$ (with $f(t) = (t - L_3(z))g(t)$),

(1'zbc) $\quad t_1(z) \leq U_1(z) < \alpha_0, \quad$ (with $f(t) = (t - L_4(z))g(t)$).

(1 zbc) and (1'zbc) present bounds $U_1(z)$ which we apply as follows:

(1 zbc) is good when $\rho_0|C| < 3$, and
(1'zbc) works well when $\rho_0|C| \geq 3$.
In fact, we define and use
$U_1(z) = \min\{U_1(z)$ from (1 zbc), $U_1(z)$ from (1'zbc)$\}$.

The improved algorithm

Case 1.1: There exists a z-bad special triple.

Denote by

$\mathcal{H} := \{h(t) \mid h(t)$ is real polynomials of degree at most $2k-2$
which are decreasing in $(-\infty, \alpha_0)$, nonnegative in $[\alpha_0, \alpha_{k-1}]$
and increasing in $(\alpha_{k-1}, +\infty)\}$.

For example, $g(t), t^{2k-2}, h(t) = (t-a_1)^2(t-a_2)^2 \ldots (t-a_{k-1})^2 \in \mathcal{H}$,
where $\alpha_0 \le a_1 \le a_2 \le \cdots \le a_{k-1} \le \alpha_{k-1}$.

Theorem 4. (x-check for existence of C)
Suppose that $h(t) \in \mathcal{H}$ is such that

$$S_x(h) := h_0 M - h(1) - 2h(U_2(x)) - h(L_{M-1}(x)) < 0.$$

Then there exist no spherical τ-designs with odd $\tau = 2k-1$, odd size M and a z-bad special triple.

A recursive procedure:
(1), (2a), (3a), (4a), (5), (6a), (1 zbc), (1'zbc), (5), (6a), x-check.

Case 1.2: There exists a y-bad special triple which is not z-bad.

Let $\{x, y, z\} \subset C$ be a y-bad special triple which is not z-bad, i.e. $t_2(z) \le \alpha_0 < t_2(y)$. We obtain consecutively:

(1 ybc) $t_1(y) \le U_1(y) < \alpha_0$, (with $f(t) = (t - L_3(y))g(t)$);

(1'ybc) $t_1(y) \le U_1(y) < \alpha_0$, (with $f(t) = (t - L_4(y))g(t)$);

(2 ybc) $t_3(z) \ge L_3(z)$, (with $g(t)$);

(3 ybc) $t_4(z) \ge L_4(z)$, (with $g(t)$);

(4 ybc) $t_1(z) \le U_1(z) < \alpha_0$, (with $f(t) = (t - L_3(z))g(t)$);

(4'ybc) $t_1(z) \le U_1(z) < \alpha_0$, (with $f(t) = (t - L_4(z))g(t)$).

The improved algorithm

Case 1.2: There exists a y-bad special triple which is not z-bad.

Theorem 5. (y-check for existence of C)
Suppose that $h(t) \in \mathcal{H}$ is such that

$$S_y(h) := h_0 M - h(1) - h(U_1(y)) - h(U_2(y)) - h(L_{M-1}(y)) < 0.$$

Then there exist no spherical τ-designs with odd $\tau = 2k-1$, odd cardinality $|C| = M$ and an y-bad special triple.

A recursive procedure:
(2b), (3b), (4b), (6b), (1 ybc), (1'ybc), (2 ybc), (3 ybc), (4 ybc), (4'ybc), (5), (6b), y-check.

We now suppose that $t_2(z) \leq \alpha_0$ and $t_2(y) \leq \alpha_0$ (opposite to the bad pairs from Definition 3) in every special triple.

Theorem 6. If there are no bad triples in C then at least one of the following holds:

(i) there exists a special triple $\{x, y, z\} \subset C$ such that $t_{|C|-2}(x) \geq 2\alpha_0^2 - 1$ and $t_{|C|-2}(z) \geq 2\alpha_0^2 - 1$,
(ii) there exists a point $x' \in C$ such that $t_3(x') \leq \alpha_0$.

The case (ii) in Theorem 6 necessarily leads to $\rho_0|C| \geq 3$.

The converse inequality was imperative in our old works.

Overcoming this is the major improvement here.

Case 2.1: There is a special triple as in Theorem 6(i).

Let $\{x, y, z\} \subset C$ be a special triple as in Theorem 6(i), i.e.

$$t_{|C|-2}(x) \geq L_{|C|-2}(x) = 2\alpha_0^2 - 1 \text{ and}$$
$$t_{|C|-2}(z) \geq L_{|C|-1}(z) = 2\alpha_0^2 - 1.$$

We obtain consecutively:

(1 xz) $\quad L_3(z) \leq t_3(z), \qquad$ (with $g(t)$);

(2 xz) $\quad L_4(z) \leq t_4(z), \qquad$ (with $g(t)$);

(3 xz) $\quad t_1(z) \leq U_1(z) < \alpha_0, \qquad$ (with $f(t) = (t - L_3(z))g(t)$);

(3'xz) $\quad t_1(z) \leq U_1(z) < \alpha_0, \qquad$ (with $f(t) = (t - L_4(z))g(t)$);

(4 xz) $\quad t_{|C|-1}(x) \geq L_{|C|-1}(x)$;

The improved algorithm

Case 2.1: There is a special triple as in Theorem 6(i).

(5 xz) $\quad t_{|C|-1}(z) \geq L_{|C|-1}(z) = \max\{\alpha_{k-1}, 2U_1^2(z) - 1\}.$

Theorem 7. (x-check for existence of C)
Suppose that $h(t) \in \mathcal{H}$ is such that

$$S_x(h) := h_0 M - h(1) - 2h(U_2(x)) - h(L_{M-2}(x)) - h(L_{M-1}(x)) < 0.$$

Then there exist no spherical τ-designs with odd $\tau = 2k - 1$, odd cardinality $|C| = M$ and a special triple as in Theorem 6(i).

A recursive procedure:
(1 xz), (2 xz), (3 xz), (3'xz), (4 xz), (5 xz), x-check.

Case 2.2: There is a point $x' \in C$ as in Theorem 6(ii).

We recall that $\rho_0 |C| \leq 4$ and this implies $t_4(x) > \alpha_0$ for every point $x \in C$. Therefore in this case we have $x' \in C$ such that

$$t_3(x') \leq \alpha_0 < t_4(x').$$

Let $y', z', u' \in C$ be such that

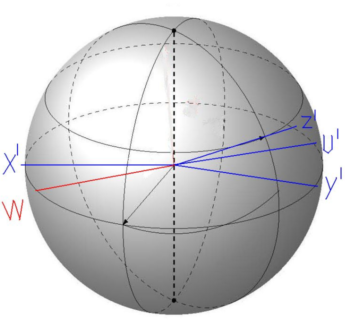
$$t_1(x') = \langle x', y' \rangle \leq t_2(x') = \langle x', z' \rangle \leq t_3(x') = \langle x', u' \rangle \leq \alpha_0.$$

Then $t_2(y') \leq \alpha_0$, $t_2(z') \leq \alpha_0$ and $t_2(u') \leq \alpha_0$.

The least two inner products of y', z', u' define at least one point $w \in C \setminus \{x', y', z', u'\}$ such that $t_2(w) \leq \alpha_0$.

The improved algorithm

Case 2.2: There is a point $x' \in C$ as in Theorem 6(ii).



$$\{x', y', z', u', w\} \subset C$$

The improved algorithm

Case 2.2: There is a point $x' \in C$ as in Theorem 6(ii).

The above implies $\langle x', w\rangle \geq 2\alpha_0^2 - 1$, $\langle y', z'\rangle \geq 2\alpha_0^2 - 1$, $\langle z', u'\rangle \geq 2\alpha_0^2 - 1$ and $\langle u', y'\rangle \geq 2\alpha_0^2 - 1$.
Moreover, the points y', z', u' are close each other and we can estimate the closest pair of them.

Lemma 8. The inner product of the closest pair from the set $\{y', z', u'\}$ is at least $\frac{3\alpha_0^2-1}{2}$.

It is clear that the worst case is when y' and z' are the closest points. Indeed, otherwise we have bounds for u' which are better than their counterparts for z'. Therefore we may assume that

$t_{|C|-2}(z') \geq L_{|C|-2}(z') = 2\alpha_0^2 - 1$ and

$$t_{|C|-1}(z') \geq L_{|C|-1}(z') = \frac{3\alpha_0^2 - 1}{2}.$$

The improved algorithm

Case 2.2: There is a point $x' \in C$ as in Theorem 6(ii).

We obtain consecutively:

(1 x') $L_3(z') \le t_3(z')$, (with $g(t)$);

(2 x') $L_4(z') \le t_4(z')$, (with $g(t)$);

(3 x') $t_1(z') \le U_1(z') < \alpha_0$, (with $f(t) = (t - L_3(z'))g(t)$);

($3'x'$) $t_1(z') \le U_1(z') < \alpha_0$, (with $f(t) = (t - L_4(z'))g(t)$);

(4 x') $t_{|C|-1}(x') \ge L_{|C|-1}(x')$;

(5 x') $t_4(x') \ge L_4(x')$, (with $g(t)$);

(6 x') $t_1(x') \le U_1(x') < \alpha_0$, (with $f(t) = (t - L_4(x'))g(t)$).

Theorem 9. (x'-check for existence of C)
Suppose that $h(t) \in \mathcal{H}$ is such that

$$S'_x(h) := h_0 M - h(1) - h(U_1(x')) - 2h(\alpha_0) - h(L_{M-1}(x')) < 0.$$

Then there exist no spherical τ-designs with odd $\tau = 2k-1$, odd cardinality $|C| = M$ and a special triple as in Theorem 5.1(ii).

Theorem 9 can be formulated

$$M \geq \frac{h(1) + h(U_1(x')) + 2h(\alpha_0) + h(L_{M-1}(x'))}{h_0}$$

as necessary existence condition (linear-programming-like bound).

The whole procedure is implemented in Maple 15.

There are 37 cases with $n \leq 50$ and $\rho_0|C| \leq 3$ (this implies that the case of Theorem 6(i) does not occur) were left open after our previous results. Our strengthening allows to rule out 21 of them.
Therefore we have the following theorem.

Theorem 10. Let $C \subset \mathbb{S}^{n-1}$, $3 \leq n \leq 50$, be a spherical 3-design with odd cardinality M. Then $\rho_0|C| > 3$ with possible exceptions in 16 cases: $(n, M) = (11, 27)$, $(15, 37)$, $(20, 49)$, $(24, 59)$, $(25, 61)$, $(29, 71)$, $(30, 73)$, $(33, 81)$, $(34, 83)$, $(38, 93)$, $(39, 95)$, $(42, 103)$, $(43, 105)$, $(44, 107)$, $(47, 115)$, $(48, 117)$.

Complete solution of Problem 2 for $\tau = 3$ in the folloing dimensions $n = 4, 6, 7, 8, 9, 10, 12, 13, 14, 16, 17, 18, 21, 22$ and 26.

Results

Spherical 5- and 7-designs with odd cardinality.

For $\tau = 5$, our strengthened approach allows calculations in more cases, namely in dimensions $3 \le n \le 50$, which were not considered earlier. Here we ruled out all but one cases with $\rho_0|C| \le 3$ (the exception is $n = 4$, $M = 23$) and some cases with $\rho_0|C| > 3$ in dimensions $3 \le n \le 50$.
Total number of new nonexistence results: 1001

For $\tau = 7$, we consider additionally to our previous results the dimensions $21 \le n \le 30$. We ruled out all but one cases with $\rho_0|C| \le 3$ (the exception is $n = 3$, $M = 23$) and some cases with $\rho_0|C| > 3$ in dimensions $3 \le n \le 30$.
Total number of new nonexistence results: 1154

The present situation with all possible cardinalities of 3-, 5- and 7-designs in small dimensions can be seen at

http://www.fmi.uni-sofia.bg/algebra/mstoyanova.shtml

Results

New asymptotic bounds

The best constants A_τ obtained by this improved algorithm.

τ	Previously best known A_τ	New value of A_τ
3	1.19625	1.21050
5	1.12286	1.12655
7	1.08496	1.08958
9	1.06491	1.07015
11	1.05251	1.05823
13	1.04409	1.05009
15	1.03799	1.04416
17	1.03337	1.03965
19	1.02976	1.03602
21	1.02685	1.03288
23	1.02446	1.03024
25	1.02246	1.02808

THANK YOU FOR YOUR ATTENTION !