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Moments of orthogonal arrays

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Orthogonal arrays

- $H(n,2)$ - binary Hamming space of dimension n.
- an *orthogonal array*, or equivalently, a τ*-design* C in $H(n,2)$ is an $M \times n$ matrix of a code C such that every $M \times \tau$ submatrix contains all ordered τ-tuples of $H(\tau,2)$, each one exactly $\frac{|C|}{2^\tau}$ times as rows.
- the maximal τ with this property is called *strength* of the array.
- we consider $H(n,2)$ with the inner product

$$\langle x,y\rangle = 1 - \frac{2d(x,y)}{n},$$

where $d(x,y)$ is the Hamming distance between x and y.

Orthogonal arrays

Definition 1.

A code $C \subset H(n,2)$ is a τ-design in $H(n,2)$ if and only if every real polynomial $f(t)$ of degree at most τ and every point $y \in H(n,2)$ satisfy

$$\sum_{x \in C} f(\langle x, y \rangle) = f_0 |C|, \qquad (1)$$

where f_0 is the first coefficient in the expansion $f(t) = \sum_{i=1}^{n} f_i Q_i^{(n)}(t)$, $Q_i^{(n)}(t)$ are the normalized Krawtchouk polynomials.

Orthogonal arrays

The identity

$$|C|f(1)+\sum_{x,y\in C,x\neq y} f(\langle x,y\rangle) = |\ C\ |^2 f_0 + \sum_{i=1}^{n} \frac{f_i}{r_i} \sum_{j=1}^{r_i} \Big(\sum_{x\in C} v_{ij}(x)\Big)^2 \qquad (2)$$

holds for every real polynomial $f(t)=\sum_{i=1}^{n} f_i Q_i^{(n)}(t)$.

- $r_i = \binom{n}{i}$
- $v_{ij}(x)$ - Boolean functions

Moments of orthogonal arrays

Definition 2.

The numbers

$$M_i = \frac{1}{r_i}\sum_{j=1}^{r_i}\left(\sum_{x\in C} v_{ij}(x)\right)^2, 1 \le i \le n,$$

are called moments of C.

- C is *OA of strength* $\tau \Leftrightarrow M_i = 0$ for $i = 1, 2, \ldots, \tau$.
- C is antipodal $\Leftrightarrow M_i = 0$ for every odd i.
- every moment M_i is a rational number whose denominator is a divisor of the LCM of all denominators of the coefficients of $Q_i(t)$.

Basic properties of the moments

Main identity

$$|C|f(1) + \sum_{x,y\in C, x\neq y} f(\langle x,y\rangle) = |\ C\ |^2 f_0 + \sum_{i=1}^{n} f_i M_i$$

Theorem 1.

Let $C \in H(n,2)$. We have $M_i = |C| + \sum_{x,y\in C,x\neq y} Q_i(\langle x,y\rangle)$, for every $i = 1, 2, \ldots, n$.

Proof.

We set $f(t) = Q_i(t)$ in *main identity* and have $f_i = 1$, $f_j = 0$ for $j \neq i$. $Q_i(1) = 1$.

□

Basic properties of the moments

- assume that $C \subset H(n,2)$ is a τ-design
- $t_j = -1 + \frac{2j}{n}$, $j = 0,1,2,\ldots,n$
- $k_j = |\{(x,y) : \langle x,y \rangle = t_j\}|$, $j = 0,1,2,\ldots,n$

Theorem 2.

Let $f(t) = \prod_{j=0}^{n-1}(t-t_j) = \sum_{i=0}^{n} f_i Q_i^{(n)}(t)$.
Then

$$\sum_{i=\tau+1}^{n} f_i M_i = |C|(f(1) - f_0|C|).$$

Basic properties of the moments

Theorem 3.

Let the polynomial $f(t)=\sum_{i=0}^{k} f_i Q_i^{(n)}(t)$ of degree $k=n-1$ or n vanishes at all but one points $t_0, t_1, t_2, \ldots, t_{n-1}$, say $f(t_j)\neq 0$. Then

$$\sum_{i=\tau+1}^{k} f_i M_i = |C|(f(1)-f_0|C|)+k_j f(t_j).$$

Example

- $n=10$, $\tau=5$, $|C|=192$
- $k_0 \in A=\{144,146,\ldots,192\}$
- $0\leq k_9\leq r$, where $r=k_0-144$.

Orthogonal arrays and spherical codes

- $H(n,2) \longrightarrow \mathbb{S}^{n-1}$: $1 \to 1/\sqrt{n}$, $0 \to -1/\sqrt{n}$ in each coordinate.
- τ-design $C \subset H(n,2) \longrightarrow \overline{C} \subset \mathbb{S}^{n-1}$

Theorem 4.

If $\tau \geq 3$ then $\overline{C}$ has at least strength 3 as a spherical design. Moreover, all moments M_i, $i = 4, 5, \ldots, \tau$, of $\overline{C}$ as a spherical design can be calculated.

Proof.

1. the first four (up to degree 3) Gegenbauer and Krawtchouk polynomials coincide
2. we set in *main identity* $f(t) = t^i$ for $i = 4, 5, \ldots, \tau$.

□

Orthogonal arrays and spherical codes

Example

Consider again the case $n = 10$, $\tau = 5$ and $|C| = 192$ - it gives a spherical $3-$design on $\mathbb{S}^9$ with moments $M_4 \approx 187,671$, $M_5 = 0$.

1. for the smallest $k_0 = 144$ we have unique solution for all $k_i, i = 1, \ldots, 9$ and this implies $M_6 \approx 389,366$, $M_7 \approx 55,4352$, $M_8 \approx 326,391$, etc.
2. for $k_0 = 192$, we obtain an antipodal spherical code with $M_i = 0$ for all odd i.

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Thank you for your attention!