Finding one of D defective elements in some group testing models.

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ACCT2012
Pomorie, June 15-21, 2012

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Notations and definitions

- $[N] := \{1, \ldots, N\}$ be the set of elements
- $\mathcal{D} \subset [N]$ be the set of defective elements
- $D = |\mathcal{D}|$ its cardinality
- $[i, j]$ the set of integers $\{x \in \mathbb{N} : i \leq x \leq j\}$

Throughout the paper we consider worst case analysis.

Classical group testing

The classical group testing problem:
find the unknown subset $\mathcal{D}$ of all defective elements in $[N]$.

For a subset $\mathcal{S} \subset [N]$ a test $t_\mathcal{S}$ is the function $t_\mathcal{S} : 2^{[N]} \to \{0, 1\}$ with

$$t_S(\mathcal{D}) = \begin{cases} 0 & , \text{ if } |\mathcal{S} \cap \mathcal{D}| = 0 \\ 1 & , \text{ otherwise.} \end{cases} \quad (1)$$

In classical group testing a strategy is called successful, if we can **uniquely determine** $\mathcal{D}$.

Adaptive and nonadaptive group tests

Strategies are called **adaptive** if the results of the first $k-1$ tests determine the kth test.

Strategies in which we choose all tests independently are called **nonadaptive**.

General group tests

Definition

Let $f_1, f_2 : [0, N] \times [0, N] \to \mathbb{R}^+$ *be two functions with* $f_1(D, S) \le f_2(D, S)$ *for all values of* D *and* S.
We call $t_S : 2^{[N]} \to \{0, 1, \{0, 1\}\}$ *a general group tests, if*

$$t_S(\mathcal{D}) = \begin{cases} 0 & , \text{ if } |\mathcal{S} \cap \mathcal{D}| < f_1(D, |\mathcal{S}|) \\ 1 & , \text{ if } |\mathcal{S} \cap \mathcal{D}| \ge f_2(D, |\mathcal{S}|) \\ \{0, 1\} & , \text{ otherwise} \\ & \text{(the result can be arbitrarily 0 or 1).} \end{cases} \quad (2)$$

For this test function denote by $n(N, D, m)$ the minimal number of tests for finding m defective elements.

Theorem

$n(N, D, 1) \geq \lceil \log(N - D + 1) \rceil$

Proof: Let us assume that we have a successful strategy s which finds a defective element with $n < \lceil \log(N - D + 1) \rceil$ tests.
Depending on the n test results we have at most 2^n different possible results for a defective element. We denote the set of these elements by $\mathcal{E}$. It holds by assumption that $|\mathcal{E}| \leq 2^n < N - D + 1$. Therefore $|[N] \backslash \mathcal{E}| > D - 1$ and there exists a set $\mathcal{F} \subset [N] \backslash \mathcal{E}$ with $|\mathcal{F}| = D$. Now we consider the case $\mathcal{D} = \mathcal{F}$. It is obvious that strategy s cannot find any defective element with n tests. □

We consider the following special cases of this test model, where $f = f_1 = f_2$ and D is known.

Threshold group testing without gap: $f(D, |\mathcal{S}|) = u$. Thus

$$t_S(\mathcal{D}) = \begin{cases} 0 & \text{, if } |\mathcal{S} \cap \mathcal{D}| < u \\ 1 & \text{, if } |\mathcal{S} \cap \mathcal{D}| \geq u \end{cases} \tag{3}$$

Group testing with density tests: $f(D, |\mathcal{S}|) = \alpha|\mathcal{S}|$ for all values. Thus

$$t_S(\mathcal{D}) = \begin{cases} 0 & \text{, if } |\mathcal{S} \cap \mathcal{D}| < \alpha|\mathcal{S}| \\ 1 & \text{, if } |\mathcal{S} \cap \mathcal{D}| \geq \alpha|\mathcal{S}|. \end{cases} \tag{4}$$

We consider for all this test functions the adaptive model with the goal of finding m (in most cases $m = 1$) defective element.

Classical test function

We assume that $0 < D < N$ is known. Our goal is to find m defective elements.
We denote by $n_{(Cla)}(N, D, m)$ the minimal number of tests (1) for finding m defective elements.

Proposition

$n_{(Cla)}(N, D, 1) \leq \lceil \log(N - D + 1) \rceil$

Proof: We give a strategy which needs $\lceil \log(N - D + 1) \rceil$ tests. We know that the set $\mathcal{S}_0 = \{D, D + 1, \ldots, N\}$ contains at least one defective element. Thus we start with the test set $\mathcal{S}_1 \subset \mathcal{S}_0$ of size $\lfloor \frac{N-D+1}{2} \rfloor$.

Classical test function

If the test is positive, then at least one defective element is in $\mathcal{S}_1$, otherwise at least one defective element is in $\mathcal{S}_0 \backslash \mathcal{S}_1$. Therefore depending on the test result we substitute $\mathcal{S}_0$ by $\mathcal{S}_1$ or $\mathcal{S}_0 \backslash \mathcal{S}_1$ and iterate the procedure. With this method we can find one defective element with $\lceil \log(N - D + 1) \rceil$ tests. $\square$

Proposition 1 together with Theorem 1 implies the following

Corollary

1. $n_{(Cla)}(N, D, 1) = \lceil \log(N - D + 1) \rceil$,
2. $n_{(Cla)}(N, D, m) \leq m \lceil \log(N - D + 1) \rceil$.

Threshold test function without gap

We denote by $n_{(Thr)}(N, D, u, m)$ the minimal number of tests (3) for finding m defective elements , if we have N elements with D defectives and $f(D, |\mathcal{S}|) = u$.

Proposition

If $D \geq u$ then $n_{(Thr)}(N, D, u, 1) \leq \lceil \log(N - D + 1) \rceil$, in case $D < u$ it is not possible to find any defective element.

Threshold test function without gap

Proof: We give a strategy which needs $\lceil \log(N - D + 1) \rceil$ tests. We partition the set of N elements into the subsets $\mathcal{I}_1 = [1, u - 1]$, $\mathcal{I}_2 = [u, N - D + u]$, and $\mathcal{I}_3 = [N - D + u + 1, N]$. In $\mathcal{I}_2$ there is of course at least one defective, because the union of the two other subsets has cardinality $D - 1$. We can find a defective element in $\mathcal{I}_2$ by the following strategy with $\lceil \log(N - D + 1) \rceil$ tests.

Threshold test function without gap

We start with the test set

$$\mathcal{S}_1 = \{1, \ldots, u-1, u, \ldots, (u-1) + \lceil \frac{m(1)}{2}(N-D+1) \rceil\},$$

where $m(1) = 1$.

Inductively, we set $m(j) = \begin{cases} 2m(j-1) - 1 & \text{if } t_{S_{j-1}}(\mathcal{D}) = 1 \\ 2m(j-1) + 1 & \text{if } t_{S_{j-1}}(\mathcal{D}) = 0, \end{cases}$

and $\mathcal{S}_j = \{1, \ldots, u-1, u, u+1, \ldots, (u-1) + \lceil \frac{m(j)}{2^j}(N-D+1) \rceil\}$.
After $\lceil \log(N-D+1) \rceil$ tests we can find an i such that $t_{[1,i]} = 1, \; t_{[1,i-1]} = 0$. Thus using this strategy we find an defective element at the position i. If $D < u$ all test results are 0 and it is not possible to find any defective element. □

Threshold test function without gap

From Theorem 1 and Proposition 2 we get the following

Theorem

$n_{(Thr)}(N, D, u, 1) = \lceil \log(N - D + 1) \rceil$, *if* $D \geq u$.

Density tests

Let $n_{(Den)}(N, D, m, \alpha)$ be the minimal number of tests (4) for finding m defective elements, if we have N elements with D defectives. In [GKPW10] the authors obtain the following bounds for $n_{(Den)}(N, D, m, \alpha)$ assuming $D \geq \alpha N$

$$
\begin{aligned}
n_{(Den)}(N, D, 1, \alpha) &\geq \log(N - D + 1). && (5)\\
n_{(Den)}(N, D, m, \alpha) &\leq \lceil \log N \rceil + \max_{N' \leq 2\frac{m}{\alpha}} n_{(Den)}(N', m, m, \alpha), && (6)\\
n_{(Den)}(N, D, 1, \alpha) &\leq \lceil \log N \rceil. && (7)
\end{aligned}
$$

Density tests

Let us define

$$s_i = \lceil \frac{2^{n-i} - 1}{1 - \alpha} \rceil$$

where $i = 1, 2, \ldots, n-1$ and $s_n = 1$.

Theorem

Let $D > \sum_{i=1}^{n} s_i - 2^n + 1$ with maximal n be fulfilled then $n_{(Den)}(N, D, 1, \alpha) = \lceil \log(N - D + 1) \rceil$.

Density tests

Idea of the proof:

We consider test sets

$$\mathcal{S}_i = \{a_i + 1, a_i + 2, \ldots, a_i + s_i\}, \quad i = 1, \ldots, n$$

where $a_1 = 0$ and

$$a_i = \begin{cases} a_{i-1} + s_{i-1} & , \text{ if } t_{S_{i-1}}(\mathcal{D}) = 0 \\ a_{i-1} & , \text{ if } t_{S_{i-1}}(\mathcal{D}) = 1. \end{cases} \tag{8}$$

Density tests

Lemma

If $t_{S_{n-j}}(\mathcal{D}) = 1$ *then we can find one defective element after n tests.*

If $t_{S_{n-j}}(\mathcal{D}) = 0$ for all j then all remaining elements are defect.

Corollary

If $D \geq \alpha N$ *then* $n_{(Den)}(N, D, 1) = \lceil \log(N - D + 1) \rceil$.

Remarks

In [K09] it is shown that for the test (1) if D is unknown one needs N tests of finding one defective element or to claim that there is no defective element.

The results of [DR02] for row-weighted cover-free codes can be used to get nonadaptive strategies for test (4) if the number of defectives are known.

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Thank you for your attention!