The maximum and the minimum sizes of complete $(n, 3)$-arcs in $PG(2, 16)$

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joint work with

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ACCT 2012

SUMMARY

1. Introduction

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2. Algorithm

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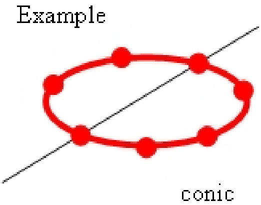
3. Results

Preliminaries

$$PG(2, q)$$

ARC : set $\mathcal{K}$ no 3-collinear points

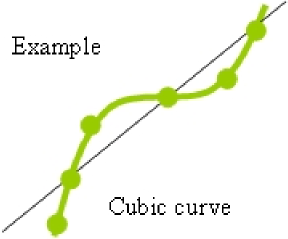


COMPLETE ARC $\mathcal{K}$: $\nexists\ \widetilde{\mathcal{K}}$ arc $\mid \mathcal{K} \not\subseteq \widetilde{\mathcal{K}},\ \ |\mathcal{K}| < |\widetilde{\mathcal{K}}|$

Preliminaries

$$PG(2, q)$$

(n,3)-arc:
set $\mathcal{K}$ no 4-collinear points
contains 3-collinear points



COMPLETE (n,3)-arc $\mathcal{K}$:
$\nexists\ \widetilde{\mathcal{K}}(n,3)-arc, \mid \mathcal{K} \not\subseteq \widetilde{\mathcal{K}}, \ \ |\mathcal{K}| < |\widetilde{\mathcal{K}}|$

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$[n, k, d]_q$ linear linear code

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NMDS codes: $s(C) = 1 = s(C^{\perp})$

NMDS codes

Theorem

A linear $[n, k, d]$code C is NMDS if and only if a generator matrix of C, say G, (and consequently each generator matrix) satisfies the following conditions:

1. *any $k-1$ columns of G are linearly independent;*
2. *there exist k linearly dependent columns in G;*
3. *any $k+1$ columns of G are of full rank*

$$[n, 3, n-3] \text{ NMDS codes} \quad \leftrightarrow \quad (n, 3)\text{−arcs of } PG(2, q)$$

S.M. Dodunekov and I. Landjev, On near-MDS codes,
J. Geometry 54 (1995), 30-43.

$(n, 3)-$arcs

Definition

$m_3(2, q)$: maximum length of an $(n, 3)-$arc of $PG(2, q)$

$t_3(2, q)$: minimum length of a complete $(n, 3)-$arc of $PG(2, q)$

Theorem

$m_3(2, q) \leq 2q + 1$, *for* $q \geq 4$

J. Thas, Some results concerning $((q + 1)(n - 1), n)$-arcs,
J. Combin. Theory Ser. A 19 (1975), 228-232.

$(n,3)-$arcs

Classification of the $(n,3)-$arcs in PG(2,q), q = 7,8,9
S. M., A. Milani, F. Pambianco Ars Comb. 2001

$m_3(3,11) = 21$, $m_3(3,13) = 23$,
S. M., A. Milani, F. Pambianco Discrete Math. 1999
S. M., A. Milani, F. Pambianco Discrete Math. 2005

Classification of the $(n,3)-$arcs in PG(2,q), q = 11,13
K. Coolsaet, H. Sticker J. Comb. Des. 2012

Algorithm

- Exhaustive search by backtracking

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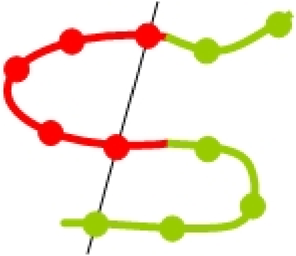
  - Size of the maximal arc contained in the $(n, 3)-$arc
  - Distribution of the points on the 2-secants of the $(n, 3)-$arc

Basic Theorem

Theorem

An $(n, 3)$-arc $\mathcal{K}$ in $PG(2, q)$, $n \geq \alpha + \binom{\alpha}{2}$, contains an arc of size $\alpha + 1$.



J. Bierbrauer, G. Faina, S. M., F. Pambianco X ACCT 2006

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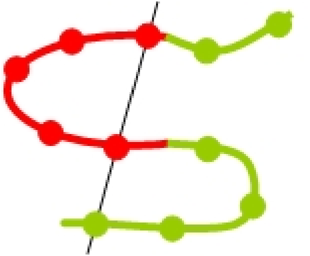

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A $(28, 3) - arc$ contains an 8-arc

Classification process

- classify all arcs of size s

$$\{A_i^s\}$$

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- data parallelism for extension process

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A^s ARC of size s

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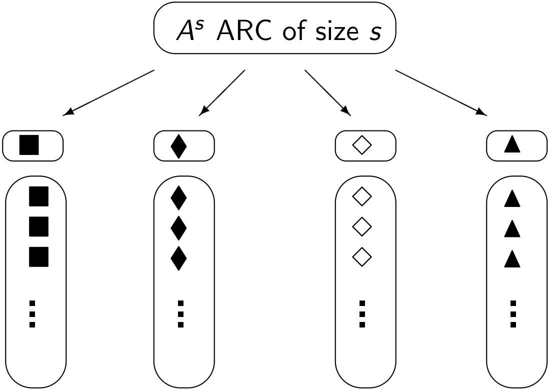

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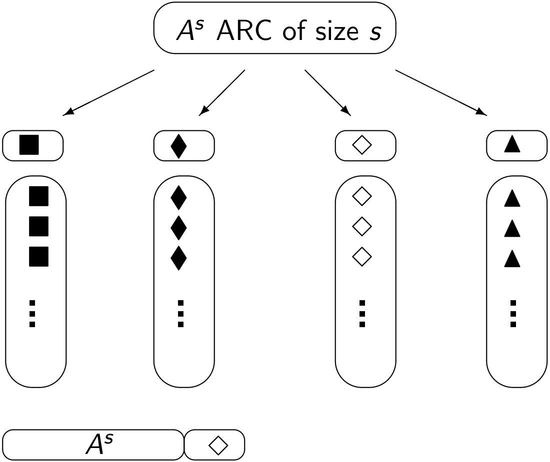

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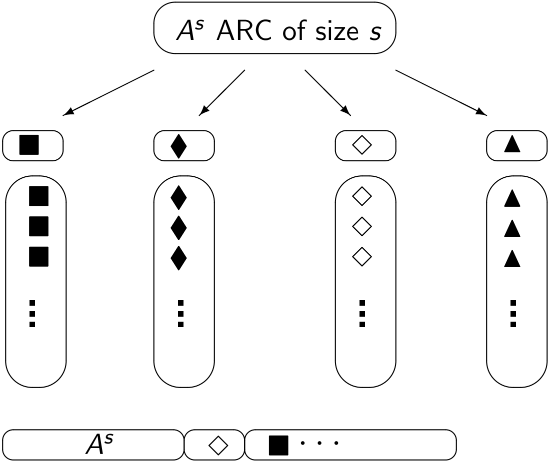

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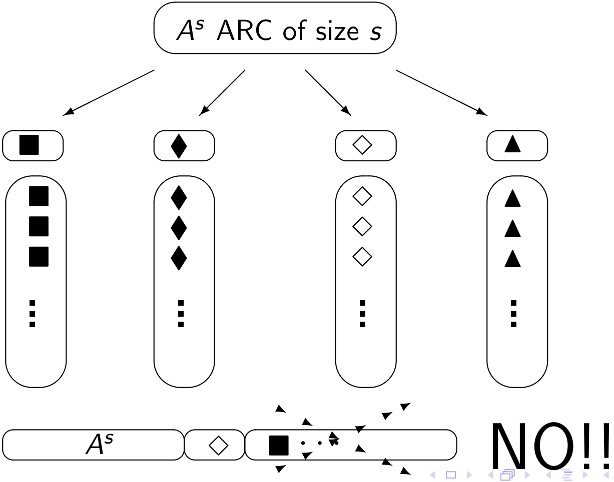

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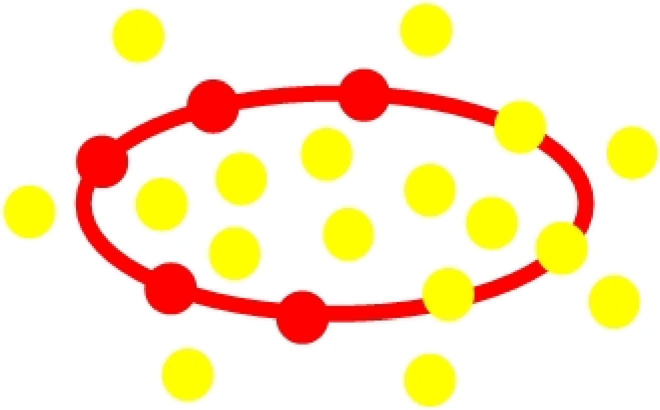
  . . .

  - compute all (29,3)-arcs containing a 8-arc but not a 9-arc

Constraints on solution: size of arc contained I



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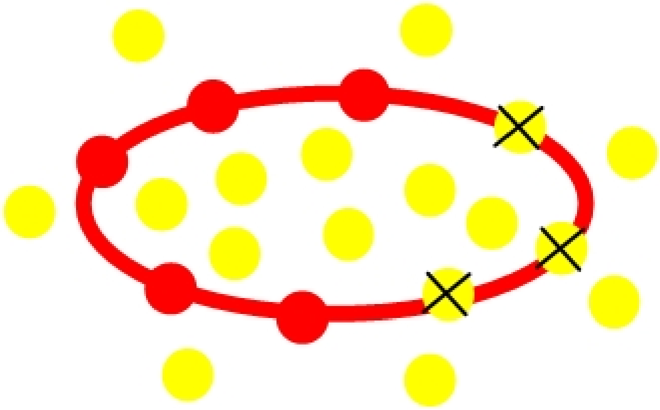

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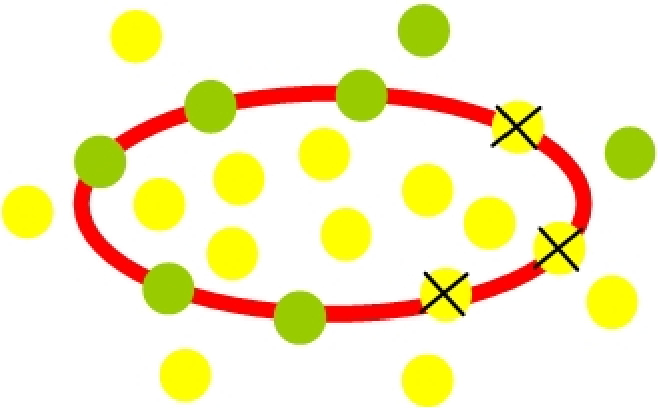

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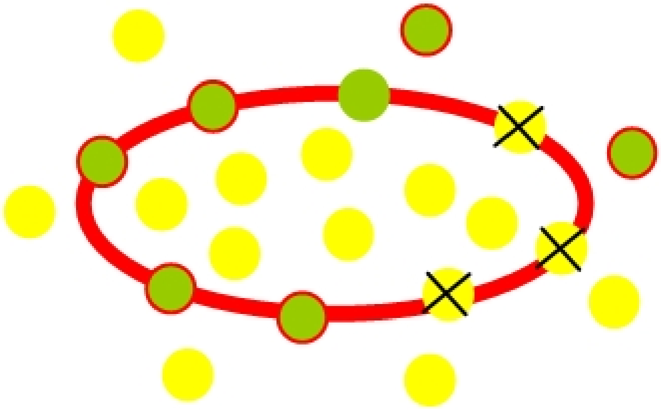


Constraints on solution: size of arc contained II

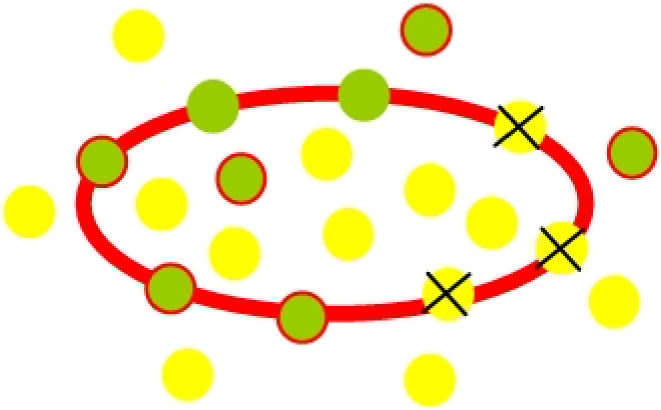
Let C' be the $(n, 3)$-arc to extend and C be the $(s, 2)$-arc contained in C'.

We collect in a table all the pairs $(P, Q), P, Q \in PG(2, q) \setminus C'$ such that:

$$\exists\, R \in C \mid (C \setminus \{R\}) \cup \{P, Q\} \text{ is an } (s+1, 2)\text{-arc.}$$

When adding a point P to the partial solution, all the points Q such that the pair (P, Q) is in the table are avoided.

Constraints on solution: size of arc contained III



Constraints on solution: size of arc contained III

Trade – off between the number of eliminated points and the computational cost of searching a sub-arc inside a $(n, 3)-$arc

At a certain level of the backtracking, random control to test if the the partial solution contains an arc too big.

In this case that branch of the search space can be pruned.

Constraints on solution: distribution of the candidates points on secants

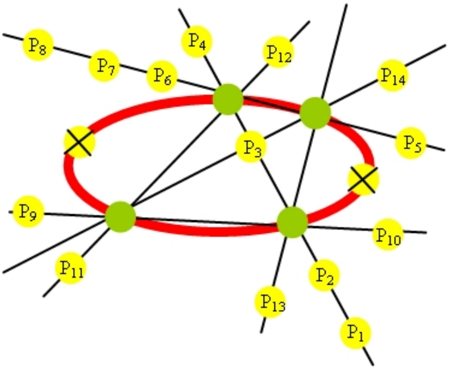
Constraint on the distribution of the candidates points on the secants of the $s-$arc that the solution $(n, 3)-$arc contains

Arcs in Finite Projective Planes, G.R. Cook, PhD Thesis, Univ. of Sussex(2011)
available on internet http://sro.sussex.ac.uk/

Constraint: distribution of candidates points on secants

To each secant, only one point can be added to obtain an $(n, 3)-$arc

Line	Points contained	β
ℓ_1	$P1,\ P2,\ P3,\ P4$	3
ℓ_2	$P5,\ P6,\ P7,\ P8$	2
ℓ_3	$P9,\ P10$	1
ℓ_4	$P11,\ P12$	0
ℓ_5	$P13$	0
ℓ_6	$P14$	0



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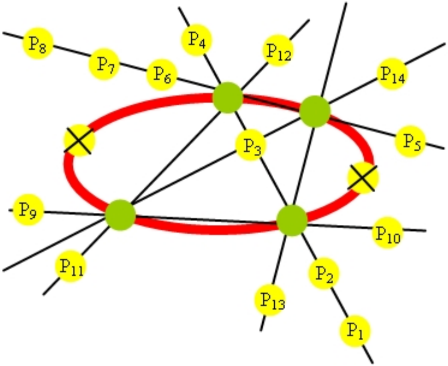

constructing an $(8, 3)-$arc choose the $i-$th points in lines β, $\beta < i$

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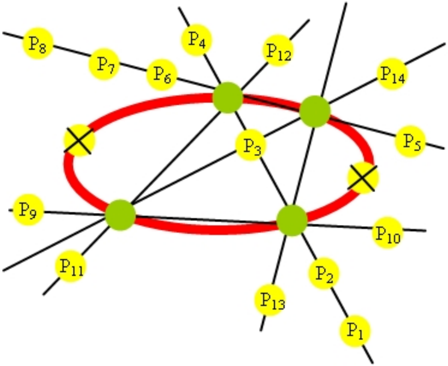

constructing an $(8, 3)-$arc choose the $i-$th points in lines β, $\beta < i$

More effective when searching for big $(n, 3)-$arcs

Results: maximal $(n, 3)-$arcs in $PG(2, 16)$

Theorem

The maximum size of complete $(n, 3)$-arcs in $PG(2, 16)$ is 28. There exists a unique $(28, 3)$-arc.

Table: Execution time of the search for $(n, 3)$-arcs in $PG(2, 16)$ with $n \geq 29$ containing an arc $\mathcal{A}$

$\|\mathcal{A}\|$	8	9	10	11	12	13	14	15	16	17
Time	2 d	22 d	20 d	4 d	2 d	1.2 h	15 m	3 m	< 1 m	$< 10s$

The complete $(28, 3)$-arc is an example of $(1, 18)$-saturating set with μ-density $\delta = 1.285714$ (see [B.D.G.M.P. ACCT2012]).

Results about NMDS codes

In the language of coding theory, the previous Theorem can be rewritten as:

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The corollary closes, for $h \leq 27$, some open cases in tables of MinT online database, http://mint.sbg.ac.at

Results about NMDS codes

Theorem

There exists no $[29, 4, 25]_{16}$ NMDS code.

Proof.

0	$[28, 3, 25]_{16}$ code
0	
0	
1	vector of weight at least 24

□

Results: smallest complete $(n, 3)-$arcs in $PG(2, 16)$

Theorem

The minimum size of complete $(n, 3)$*-arcs in* $PG(2, 16)$ *is* 15.
There exists a unique complete $(15, 3)$*-arc in* $PG(2, 16)$.
It contains a $(9, 2)$*-arc, but not a greater arc.*

Proof.

Exhaustive search for $(n, 3)-$arcs with $n \leq 14$.
It lasted 11 days on a 3.2 Ghz CPU.
Classification lasted 45 days on a 3.2 Ghz CPU. □

Results: smallest complete $(n, 3)-$arcs in $PG(2, 16)$

Table: The complete $(15, 3)$-arc in $PG(2, 16)$

Points	ℓ_0	ℓ_1	ℓ_2	ℓ_3	G
1 0 0 1 1 1 1 1 1 1 1 1 1 1 1 0 1 0 1 0 1 2 2 4 9 9 11 11 13 13 0 0 1 1 11 8 5 10 10 2 8 2 11 1 12	92	138	12	31	$\mathcal{S}_3$

$GF(16) = \{0, 1 = \alpha^0, 2 = \alpha^1, \ldots, 15 = \alpha^{14}\}$, α primitive element such that $\alpha^4 + \alpha^3 + 1 = 0$.

Results in $PG(2,17)$

Theorem

There exist no complete $(n,3)$-arcs in $PG(2,17)$, with $n > 28$ containing an arc of size greater than 12.

Theorem

The smallest size of complete $(n,3)$-arcs in $PG(2,17)$ is at most 18. There exist no complete $(n,3)$-arcs in $PG(2,17)$, with $n \leq 17$ containing an arc of size less than 8.

Results in $PG(2,19)$

Theorem

The maximum size of complete $(n,3)$-arcs in $PG(2,19)$ is at least 31. There exist no complete $(n,3)$-arcs in $PG(2,19)$, with $n \geq 31$ containing an arc of size greater than 14.

Theorem

The smallest size of complete $(n,3)$-arcs in $PG(2,19)$ is at most 20.

THANKS FOR THE ATTENTION!