New type of estimate and new upper bounds for the smallest size of complete arcs in $PG(2, q)$

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OUTLINE

- Introduction about complete arcs in projective planes
- Results with greedy algorithms
- Results with FOP algorithm

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Let $PG(2, q)$ be the projective plane over $\mathbb{F}_q$.

Definition

A k-arc in $PG(2, q)$ is a set of k points no three of which are collinear.
A k-arc is *complete* if it is not contained in a $(k + 1)$-arc of $PG(2, q)$.

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$$\text{n-arc } \mathcal{K} = \{P_1, P_2, \ldots, P_n\}$$
$$P_i = (x_0^i, x_1^i, x_2^i)$$

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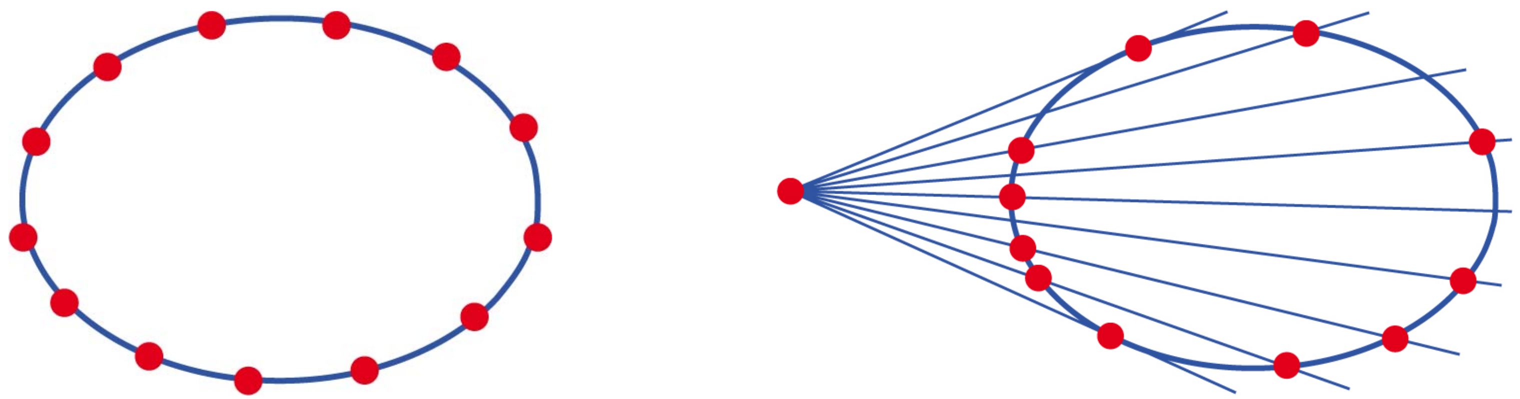
$$\begin{array}{c}\text{Complete} \\ n-\text{arcs}\end{array} \xrightarrow[\textit{to}]{\textit{correspond}} \begin{array}{c}[n, n-3, 4] \text{ MDS codes} \\ \text{covering radius} = 2\end{array}$$

Maximum size

$m_2(2,q) =$ maximum size of complete arcs in $PG(2,q)$



$$m_2(2,q) = \begin{cases} q+1 & q \text{ odd} \\ q+2 & q \text{ even} \end{cases}$$

Minimum size

$t_2(2,q) =$ minimum size of complete arcs in $PG(2,q)$

q	$t_2(2,q)$	q	$t_2(2,q)$
2, 3	4	17, 19, 23	10
4, 5, 7, 8, 9	6	25, 27	12
11	7	29	13
13	8	31, 32	14
16	9		

$$t_2(2,q) > \begin{cases} \sqrt{2q}+1 & \forall\ q \\ \sqrt{3q}+\frac{1}{2} & q = p^h,\ h = 1,2,3 \end{cases}$$

$\bar{t}_2(2,q) =$ minimum known size of complete arcs in $PG(2,q)$

Theorem

If $q \leq 841$ *or* $q \in \{31^2, 2^{10}, 37^2, 41^2, 7^4\}$, *then* $\bar{t}_2(2,q) < 4\sqrt{q}$.

A.A. Davydov, G. Faina, S. Marcugini, F. Pambianco, *On sizes of complete caps in projective spaces* $PG(n,q)$ *and arcs in planes* $PG(2,q)$, J. Geom. 94 (2009) 31-58.

Theorem

In $PG(2,q)$, *the following holds.*

$$\begin{aligned} t_2(2,q) &< 4.5\sqrt{q} \quad \text{for } q \leq 2503,\ q \in Q_1. \\ t_2(2,q) &< 4.73\sqrt{q} \quad \text{for } q \leq 4397. \\ t_2(2,q) &< 4.98\sqrt{q} \quad \text{for } q \in T_2. \end{aligned}$$

$$T_2 = \{4447, 4451, 4457, 4463, 4481, 4483, 4493, 4507, 4513, 4517, 4523, 5003, 5347, 5641, 5843, 6011, 8192\}.$$

D. B., A.A. Davydov, G. Faina, S. Marcugini, F. Pambianco, *On sizes of complete arcs in* $PG(2,q)$, Discrete Math. 312 (2012), 680-698.

$\boxed{t(\mathcal{P}_q)}$: size of the smallest complete arc in the projective plane $\mathcal{P}_q$ (not necessary Galois) of order q.

$$\boxed{\begin{array}{c} t(\mathcal{P}_q) \leq d\sqrt{q}\log^c q,\ c \leq 300 \\ c, d \text{ absolute constants} \end{array}}$$

J.H. Kim, V. Vu, *Small complete arcs in projective planes*, Combinatorica 23 (2003) 311-363.

PROBABILISTIC METHODS

Greedy Algorithm

Theorem

$$
\begin{array}{lllll}
t_2(2,q) & < & \sqrt{q}\ln^{0.75} q & \text{for } 23 \le q \le 5107 & \textbf{[1]} \\
t_2(2,q) & < & \sqrt{q}\ln^{0.75} q & \text{for } 23 \le q \le 9109 & \textbf{[2]} \\
t_2(2,q) & < & \sqrt{q}\ln^{0.73} q & \text{for } 109 \le q \le 10111 & \textbf{[3]}
\end{array}
$$

[1] D. Bartoli, A. A. Davydov, G. Faina, S. Marcugini, and F. Pambianco, On sizes of complete arcs in $PG(2,q)$, *Discrete Math.* **312** (2012), 680-698.

[2] D. Bartoli, A. A. Davydov, G. Faina, S. Marcugini, and F. Pambianco, Upper bounds on the smallest size of a complete arc in the plane $PG(2,q)$, http://arxiv.org/abs/1111.3403.

[3] D. Bartoli, A. A. Davydov, G. Faina, S. Marcugini, and F. Pambianco, *New upper bounds on the smallest size of a complete arc in the plane* $PG(2,q)$, Proc. XIII Int. Workshop on Algebraic and Combin. Coding Theory, ACCT2012, Pomoria, Bulgaria, 2012.

Greedy Algorithm

$G = \{11003, 11971, 11981, 11987, 12007, 13001, 14009, 15013, 16001, 2^{14}, 17011, 18013, 19001, 20011, 21001, 22003, 23011, 24001, 25013, 26003, 27011, 28001, 29009, 30011, 31013, 32003, 2^{15}, 33013, 34019, 35023, 36007, 37003, 38011, 39019, 40009, 41011, 42013, 43003, 44017, 2^{18}\}.$

Theorem

In $PG(2, q)$, the following holds.

$$
\begin{array}{lll}
t_2(2,q) & < & \sqrt{q}\ln^{0.73} q \quad \text{for } 109 \le q \le 10111,\ q \in G. \\
t_2(2,q) & < & 3.8\sqrt{q} \quad \text{for } q \le 541,\ q = 601, 661. \\
\bar{t}_2(2,q) & < & 4\sqrt{q} \quad \text{for } q \le 841,\ q = 857, 31^2, 2^{10}, 37^2, 41^2, 7^4. \\
t_2(2,q) & < & 4.3\sqrt{q} \quad \text{for } q \le 1627,\ q = 1657, 1663, 41^2, 1697, 7^4. \\
t_2(2,q) & < & 4.5\sqrt{q} \quad \text{for } q \le 2647,\ q = 2659, 2663, 2683, 2693, 2753, 2801. \\
t_2(2,q) & < & 4.8\sqrt{q} \quad \text{for } q \le 5419,\ q = 5441, 5443, 5471, 5483, 5501, 5521. \\
t_2(2,q) & < & 5\sqrt{q} \quad \text{for } q \le 9497,\ q = 9539, 9587, 9613, 9649, 9689, 9973. \\
t_2(2,q) & < & 5.04\sqrt{q} \quad \text{for } q \le 10111. \\
t_2(2,q) & < & 5.5\sqrt{q} \quad \text{for } q \le 10111,\ q \in G \text{ with } q \le 40009.
\end{array}
$$

Greedy Algorithm

Conjecture

In $PG(2,q)$ it holds that

$$\begin{aligned} t_2(2,q) &< \sqrt{q}\ln^{0.73} q && \textit{for all } q \geq 109. \\ t_2(2,q) &< 5.5\sqrt{q} && \textit{for } q \leq 40009. \end{aligned}$$

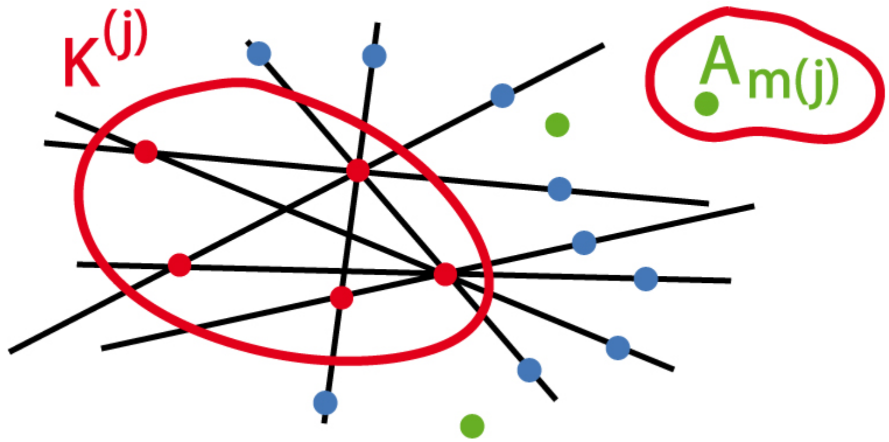
Fixed Order of Points

Assumptions:

1. In $PG(2, q)$, there are relatively many complete k-arcs with size of order $k \approx \sqrt{q} \ln q$,
2. in $PG(2, q)$, a random complete k-arc has the size of order $k \approx \sqrt{q} \ln q$ with high probability;
3. the sizes of complete arcs obtained by Algorithm FOP vary insignificantly with the respect to the order of points.

Algorithm

$$PG(2,q) := \{A_1, A_2, \ldots, A_{q^2+q+1}\}$$



$$K^{(0)} := \emptyset \quad K^{(1)} := \{P_1\} \quad K^{(2)} := \{P_1, P_2\}$$

$$K^{(j+1)} = K^{(j)} \cup \{A_{m(j)}\}$$

$m(j)$: minimum index s.t. $A_{m(j)}$ is not saturated by $K^{(j)}$

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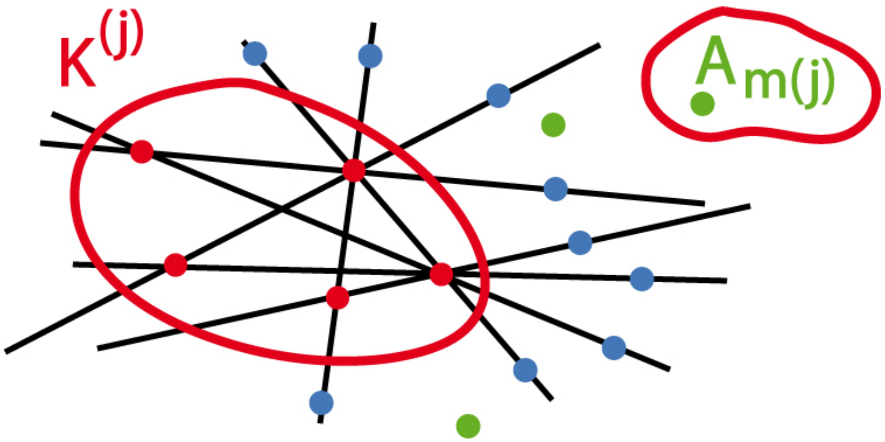

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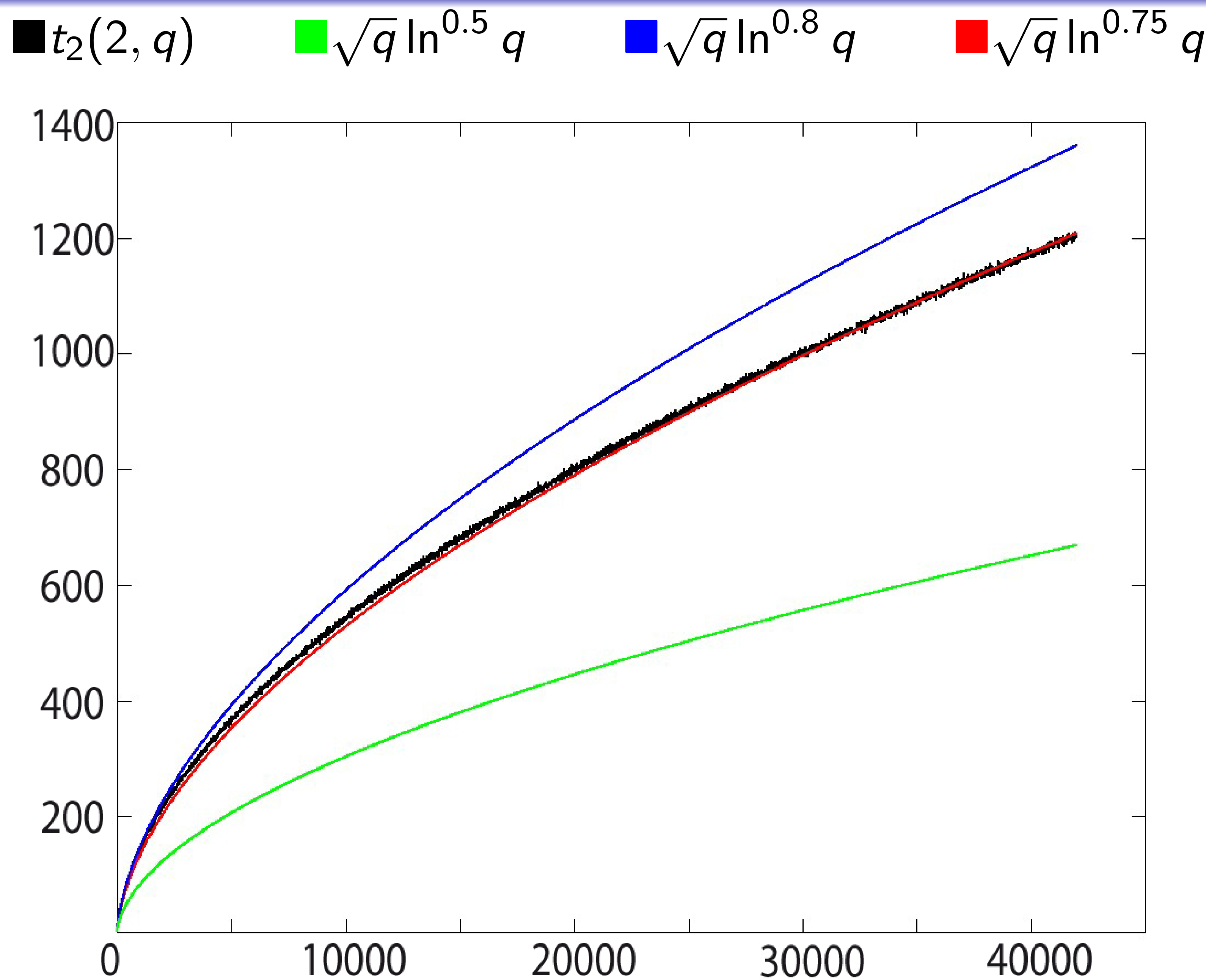
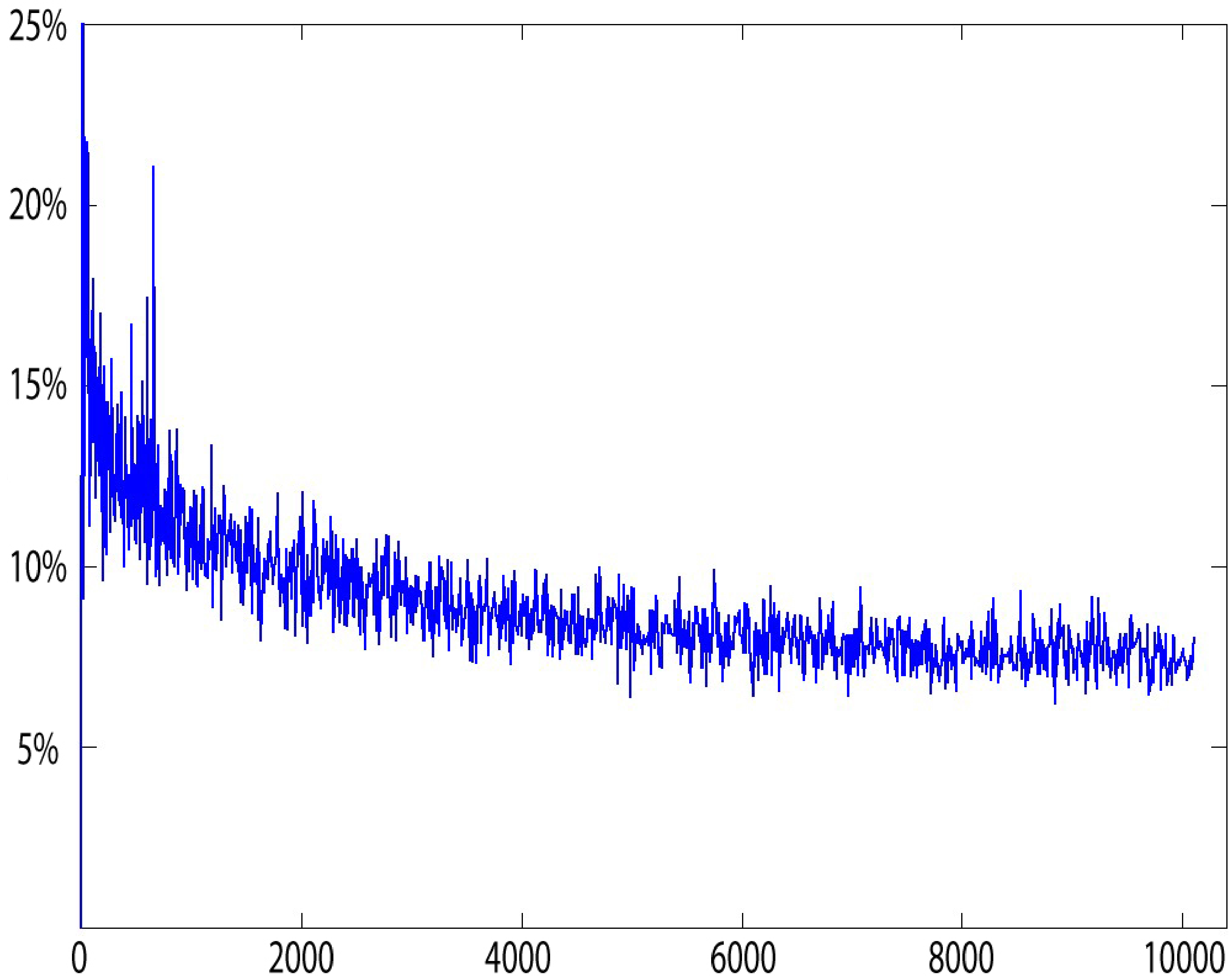
LEXICOGRAPHICAL SINGER

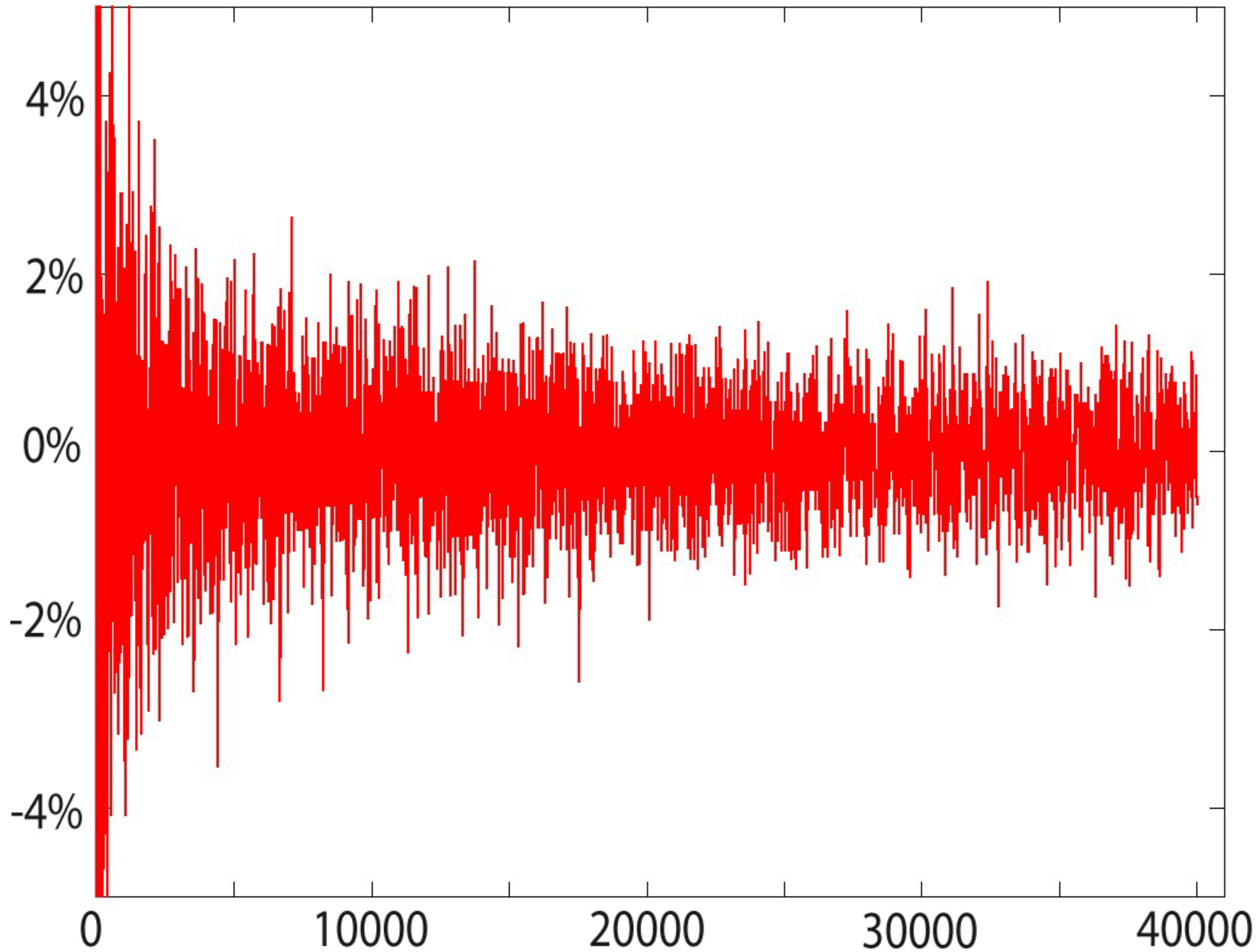
Results with Lexicographical order $q \leq 42009$



Comparison between Lexicographical and Greedy



Comparison in percentage between Singer and Lexicographical



$$R = \{43003, 44017, 45007, 46021, 47017, 48017, 49009, 50021, 51001, 52009, 53003,$$

$$54001, 55001, 56003, 57037, 58013, 59009, 60013, 69997, 70001, 79999, 80021\}$$

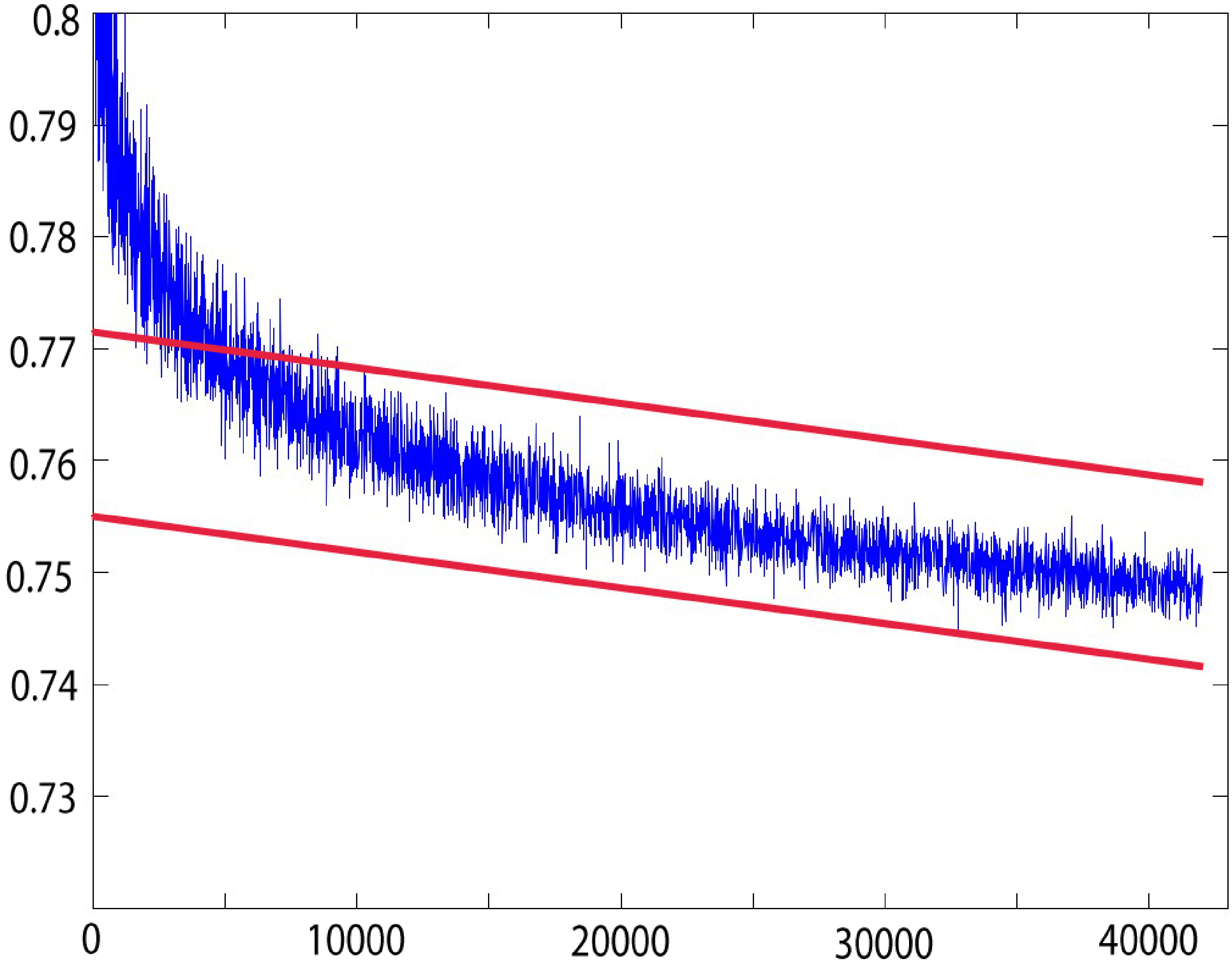
Theorem

α, β : absolute positive constants independent of q.

$$t_2(2,q) < \sqrt{q}\log^{c(q)} q, \;\; c(q) < \alpha - \beta q$$

$$\alpha = 0.7715, \;\; \beta = 3.2 \cdot 10^{-7}$$

where $9311 \le q \le 42013$, $q \in R$, q *prime.*


0.8
0.79
0.78
0.77
0.76
0.75
0.74
0.73
0
10000
20000
30000
40000

Some values of $\bar{t}_2^L = \bar{t}_2^L(2,q)$ for $q \geq 42013$, q prime

q	$\bar{t}_2^L$	$\alpha_1 - \beta q$	$c_L(q)$	$\alpha_2 - \beta q$	q	$\bar{t}_2^L$	$\alpha_1 - \beta q$	$c_L(q)$	$\alpha_2 - \beta q$
42013	1207	0.7580	0.7496	0, 7415	43003	1218	0.7577	0.7478	0.7412
44017	1238	0.7574	0.7491	0.7409	45007	1250	0.7570	0.7478	0.7405
46021	1265	0.7567	0.7475	0.7402	47017	1291	0.7564	0.7509	0.7399
48017	1296	0.7561	0.7475	0.7396	49009	1316	0.7558	0.7490	0.7393
50021	1328	0.7554	0.7480	0.7389	51001	1339	0.7551	0.7468	0.7386
52009	1357	0.7548	0.7477	0.7383	53003	1364	0.7545	0.7454	0.7380
54001	1381	0.7542	0.7461	0.7377	55001	1403	0.7539	0.7484	0.7374
56003	1412	0.7535	0.7467	0.7370	57037	1430	0.7532	0.7477	0.7367
58013	1433	0.7529	0.7445	0.7364	59009	1448	0.7526	0.7449	0.7361
60013	1470	0.7522	0.7471	0.7357	69997	1595	0.7491	0.7448	0.7326
70001	1599	0.7491	0.7458	0.7326	79999	1707	0.7459	0.7416	0.7294
80021	1715	0.7458	0.7434	0.7293					

Conjecture

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The upper bound

$$t_2(2, q) < \sqrt{q} \log^{c(q)} q, \;\; c(q) < \alpha - \beta q$$
$$\alpha = 0.7715, \;\; \beta = 3.2 \cdot 10^{-7}$$

holds for all $q \geq 9311$.

Thank

for your attention!