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Distance regular colorings of n-dimensional rectangular grid [1]

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Abstract. A perfect coloring is defined by the following property: any two vertices of the same color have the same chromatic composition of the environment. A distance regular coloring is the perfect coloring with respect to a the completely regular code. It is shown that every irreducible distance regular coloring of the n-dimensional rectangular grid (this graph is not distance regular) has no more than $2n+1$ colors. It is proved that the minimal distance of a completely regular code in the n-dimensional rectangular grid is no more than 4.

1 Introduction

A k-coloring of graph vertices can be presented as a function φ over graph vertices with values in the set $\{1, 2, \dots, k\}$ and as a partition $\{C_1, C_2, \dots, C_k\}$ of graph vertices, where $C_i = \{\mathbf{x} \ : \ \varphi(\mathbf{x}) = i\}$, $i = 1, 2, \dots, k$. We do not distinguish between two interpretations.

A coloring with the colors $1, 2, \dots, k$ is perfect (in other terms, the partition is equitable) with the parameter matrix $A = (\alpha_{ij})_{k\times k}$ if any vertex of the color i has α_{ij} adjacent vertices of color j, $i, j = 1, 2, ..., k$.

A perfect coloring is distance regular if there exists an order of colors such that the parameter matrix became three-diagonal. In what follows we suppose that the colors are in this order. In other words, the perfect coloring is distance regular if the set of vertices of the color i, $i = 2, 3, \dots, k$, consists of all vertices at distance $i-1$ from the set of first color vertices. This notion is intimately related to the notion of completely regular code. Actually, according to the definition of the completely regular code the vertices of the first color (the last color) give the completely regular code.

Distance regular colorings are very useful in the study of invariant property of perfect structures. Notice that in a distance regular graph the distance partition with respect to an arbitrary vertex is the perfect coloring, its parameters does not depend on

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the choice of the vertex. Completely regular codes in distance regular graphs are extensively investigated. We study distance regular colorings in the graph of n-dimensional rectangular grid that is not distance regular. First we get the monotonicity of elements of the parameter matrix of the coloring (Theorem 1). Then we prove that the number of colors does not exceed $2n+1$ and show that this bound is attainable (Theorem 2). We obtain that the distance of a completely regular code is 4 or less (Theorem 3).

Denote nonzero elements of parameter matrix of the distance regular coloring:
$d_i = \alpha_{i,i-1}$ $(i = 2, 3, \ldots, k)$ – the down degree of the i-th color;
$k_i = \alpha_{i,i}$ $(i = 1, 2, \ldots, k)$ – the inner degree of the i-th color;
$u_i = \alpha_{i,i+1}$ $(i = 1, 2, \ldots, k-1)$ – the upper degree of the i-th color.
In this terms, any vertex of the color i "sees" d_i vertices of the color $i-1$, k_i vertices of color i and u_i vertices of the color $i+1$. We will say that the color i has the degree triple (d_i, k_i, u_i).

Fix an arbitrary distance regular coloring φ of $\mathbf{Z}^n$ and an arbitrary vertex $\mathbf{x} \in \mathbf{Z}^n$, $\varphi(\mathbf{x}) = i$. Let us introduce the following notation

$$D(\mathbf{x}) = \{\mathbf{y} - \mathbf{x} \ : \ \varphi(\mathbf{y}) = i-1, \ \mathbf{y} \in S(\mathbf{x})\},$$

$$I(\mathbf{x}) = \{\mathbf{y} - \mathbf{x} \ : \ \varphi(\mathbf{y}) = i, \ \mathbf{y} \in S(\mathbf{x})\},$$

$$U(\mathbf{x}) = \{\mathbf{y} - \mathbf{x} \ : \ \varphi(\mathbf{y}) = i+1, \ \mathbf{y} \in S(\mathbf{x})\},$$

where $S(\mathbf{x})$ denotes the sphere of radius 1 centered in $\mathbf{x}$. We refer the vectors in the sets $D(\mathbf{x}), I(\mathbf{x}), U(\mathbf{x})$ to as downward, inward and upward directions of the vertex $\mathbf{x}$ with respect to the coloring φ. We use the notation $-L = \{-l \ : \ l \in L\}$ for any set L of directions.

2 Reducible colorings

First consider colorings of 1-dimensional space $\mathbf{Z}^1$. For an arbitrary k there exist only three nonequivalent (up to shifts) distance regular k-colorings and they are periodical. Write theirs periods as sequences of colors:

$$1, 2, \ldots, k-1, k, k-1, \ldots, 2;$$

$$1, 1, 2, \ldots, k-1, k, k-1, \ldots, 2;$$

$$1, 1, 2, \ldots, k-1, k, k, k-1, \ldots, 2.$$

The coloring $\varphi = \varphi(x_1, \ldots, x_n)$ of $\mathbf{Z}^n$ is called reducible if it can be reduced to the 1-dimensional coloring, i.e. for any $(x_1, x_2, \ldots, x_n) \in \mathbf{Z}^n$

$$\varphi(x_1, x_2, \ldots, x_n) = \varphi_1\left(\delta_1 x_1 + \delta_2 x_2 + \ldots + \delta_n x_n\right),$$

where φ_1 is a k-coloring of $\mathbf{Z}^1$ and $\delta_1, \ldots, \delta_k \in \{0, 1, -1\}$. The parameter matrix of the coloring is referred to as reducible if it admits the reducible coloring. Write the reducible matrices:

$$\begin{pmatrix} n-\varepsilon_1 r & 2r & . & . & . & 0 & 0 \\ r & n-2r & r & . & . & . & 0 \\ . & . & . & . & . & . & . \\ 0 & . & . & . & r & n-2r & r \\ 0 & 0 & . & . & . & 2r & n-\varepsilon_2 r \end{pmatrix},$$

where r equals to the number of nonzero coefficients δ_i and $\varepsilon_1, \varepsilon_2 \in \{1, 2\}$ (the colorings with $(\varepsilon_1, \varepsilon_2) = (1, 2)$ and $(2, 1)$ are equivalent).

3 Upper and down degrees

Fix an arbitrary distance regular k-coloring of $\mathbf{Z}^n$.

Lemma 1. *For any i, $1 \leq i \leq k-1$, and any two adjacent vertices $\mathbf{x}$ and $\mathbf{y}$ of colors i and $i+1$ respectively we have $D(\mathbf{x}) \subseteq D(\mathbf{y})$ and $U(\mathbf{x}) \supseteq U(\mathbf{y})$.*

Proof. Let $l \in D(\mathbf{x})$, i.e. the color of vertex $\mathbf{z} = \mathbf{x} + l$ is $i-1$. Since the coloring is distance regular the vertex $\mathbf{v} \in \mathbf{y} + l$ should have the color i and then $l \in D(\mathbf{y})$. □

The sequence of vertices $\mathbf{x^1}, \mathbf{x^2}, \ldots, \mathbf{x^r} \in \mathbf{Z}^n$ $(r \leq k)$ is called an ascending chain, if $\varphi(\mathbf{x^i}) = \varphi(\mathbf{x^{i-1}}) + 1$ and distance between $\mathbf{x^{i-1}}$ and $\mathbf{x^i}$ equals 1 for any $i = 2, \ldots, r$. As the simple consequence of Lemma 1 we get

Corollary 1. *Let $r \leq k$ and $\mathbf{x^1}, \mathbf{x^2}, \ldots, \mathbf{x^k}$ be an ascending chain. Then*

$$D(\mathbf{x^1}) \subseteq D(\mathbf{x^2}) \subseteq \ldots \subseteq D(\mathbf{x^r}),$$

$$U(\mathbf{x^1}) \supseteq U(\mathbf{x^2}) \supseteq \ldots \supseteq U(\mathbf{x^r}).$$

So we have the monotonicity of down degrees and upper degrees:

Theorem 1. *For an arbitrary distance regular k-coloring of $\mathbf{Z}^n$*

$$d_2 \geq \ldots \geq d_{k-1} \geq d_k \quad \text{and} \quad u_1 \leq u_2 \leq \ldots \leq u_{k-1}.$$

Lemma 2. *a) If $d_i = d_{i+1}$ then $u_i \leq d_i$ and $-U(\mathbf{x}) \subseteq D(\mathbf{x})$ for any i-colored vertex $\mathbf{x}$.*
b) If $u_i = u_{i-1}$ then $d_i \leq u_i$ and $-D(\mathbf{x}) \subseteq U(\mathbf{x})$ for any i-colored vertex $\mathbf{x}$.

Proof. a) Let $l \in U(\mathbf{x})$. Then the vertex $\mathbf{x} + l$ has the color $i+1$. Because $d_i = d_{i+1}$, it follows from Lemma 1 that $D(\mathbf{x}) = D(\mathbf{x}) + l$. Hence $-l \in D(\mathbf{x}) + l = D(\mathbf{x})$. □

4 The colors with the same degree triples

It is obvious that if the degree triples of colors $i \neq j$ coincide then they are the same for all colors between i and j.

Lemma 3. *Let the colors i and $i+1$ have the same degree triples. Then for any two adjacent vertices $\mathbf{x}$ and $\mathbf{y}$ of colors i and $i+1$ respectively we have*

$$D(\mathbf{x}) = D(\mathbf{y}) = -U(\mathbf{x}) = -U(\mathbf{y}),$$

$$I(\mathbf{x}) = I(\mathbf{y}) = -I(\mathbf{x}) = -I(\mathbf{y}).$$

Proof. Consider an arbitrary direction $l \in D(\mathbf{x})$. Then by Lemma 1 $l \in D(\mathbf{y})$ and it means that $-l \in U(\mathbf{y}+l)$. Re-using Lemma 1 and with our condition $u_i = u_{i+1}$ get $-l \in U(\mathbf{y})$ and once again by Lemma 1 $-l \in U(\mathbf{x})$.

Then consider an arbitrary direction $l \in U(\mathbf{x})$, i.e. $-l \in D(\mathbf{x}+l)$ where the color of $\mathbf{x}+l$ is equal to $i+1$. Combining our condition $d_i = d_{i+1}$ with Lemma 1 get $-l \in D(\mathbf{x})$. □

Note that according to this lemma two opposite directions belongs or does not belong to the set of inner directions simultaneously. In particular, it means that in our case all inner degrees are even.

Describe the set of i-colored vertices. For any set of vertices $V \subseteq \mathbf{Z}^n$ denote dy $G(V)$ the subgraph of $\mathbf{Z}^n$ generated by the set V of vertices.

Lemma 4. *Let the colors i and $i+1$ have the same degree triples. Then for any two vertices $\mathbf{x}$ and $\mathbf{y}$ of the connected component of the graph $G(C_i)$ we have*

$$D(\mathbf{x}) = D(\mathbf{y}) = -U(\mathbf{x}) = -U(\mathbf{y}),$$

$$I(\mathbf{x}) = I(\mathbf{y}) = -I(\mathbf{x}) = -I(\mathbf{y}).$$

Proof. Show that the second equation is true for any two adjacent vertices $\mathbf{x}$ and $\mathbf{y}$ of the color i. Suppose that for some $l \in I(\mathbf{x})$ this direction is not inward for $\mathbf{y}$, i.e. $l \in U(\mathbf{x})$. Fig. 1 with the notations of vertices and Fig. 2 with theirs colors illustrate our reasoning. We reconstruct the colors in alphabetical order of vertices in Fig. 1 and get the coloring as in Fig. 2.

	c	e	g
a	x	y	b
	d	f	

Fig. 1

	i	$i+1$	$i+2$
i	i	i	i
	i	$i-1$	

Fig. 2

Here we get the contradiction for the pare of vertices $\mathbf{b}$ and $\mathbf{g}$. The first equation follows from the second and Lemma 3. □

Combining Lemma 3 and Lemma 4 we obtain

Corollary 2. *Let the colors i and $i+1$ have the same degree triples. Then for any two vertices $\mathbf{x}$ and $\mathbf{y}$ of the connected component of the graph $G(C_i \bigcup C_{i+1})$ we have*

$$D(\mathbf{x}) = D(\mathbf{y}) = -U(\mathbf{x}) = -U(\mathbf{y}),$$

$$I(\mathbf{x}) = I(\mathbf{y}) = -I(\mathbf{x}) = -I(\mathbf{y}).$$

Now we can describe the structure of connected components of the graph $G(C_i \bigcup C_{i+1})$ in case when the corresponding degree triples coincide.

Let $\delta = (\delta_1, \delta_2, \ldots, \delta_n),\ \delta_1, \delta_2, \ldots, \delta_n \in \{0, 1, -1\}$ and $c \in \mathbf{Z}$. Denote

$$M(c) = \{(x_1, x_2, \ldots, x_n) \in \mathbf{Z}^n \ :\ \delta_1 x_1 + \delta_2 x_2 + \ldots + \delta_n x_n = c\}\,.$$

Lemma 5. *Let the colors i and $i+1$ have the same degree triples and G' be a connected component of $G(C_i \bigcup C_{i+1})$. Then there exist integer c and $(0, 1, -1)$-valued vector δ such that*

$$C_i \bigcap G' = M(\delta, c), \qquad C_{i+1} \bigcap G' = M(\delta, c+1).$$

Proof. Consider the vertex $\mathbf{v} = (v_1, v_2, \ldots, v_n) \in G'$ of color i. Without loss of generality we take

$$U(\mathbf{v}) = \{\mathbf{e^1}, \ldots, \mathbf{e^s}, -\mathbf{e^{s+1}}, \ldots, -\mathbf{e^t}\},$$
$$I(\mathbf{v}) = \{\pm\mathbf{e^{t+1}}, \ldots, \pm\mathbf{e^n}\},$$

where $\mathbf{e^i},\ i = 1, \ldots, n,$ are the unit vectors. Then put the following constants:
$\delta_1 = \ldots = \delta_s = 1$,
$\delta_{s+1} = \ldots = \delta_t = -1$,
$\delta_{t+1} = \ldots = \delta_n = 0$,
$c = v_1 + \ldots + v_s - v_{s+1} - \ldots - v_t$.
We still have the formula to easily check. □

Corollary 3. *Let $\varphi : \mathbf{Z}^n \longrightarrow \{1, 2, \ldots, k\}$ be the distance regular coloring and for some $i,\ 2 \le i \le k-2$, the colors i and $i+1$ have the same degree triples. Then the degree triples are coincide for all colors from 2 to $k-1$ and the coloring is reducible.*

5 The number of colors

Consider an arbitrary distance regular coloring of $\mathbf{Z}^n$. The color j is called medial if $d_i \le u_i$ for any $i \le j$ and $d_i > u_i$ for any $i > j$. Note that the medial color exists by virtue of monotonicity of the sequences of down degrees and upper degrees (Lemma 1).

Lemma 6. *Let the distance regular coloring be irreducible and the color j is medial. Then no two of $d_2, \ldots, d_j$ are the same and no two of $u_{j+1}, \ldots, d_{k-1}$ are the same.*

Proof. Follows from Lemma 2. □

Theorem 2. *If φ is the irreducible distance regular k-coloring of n-dimensional rectangular grid $\mathbf{Z}^n$ then $k \le 2n+1$. Moreover, there exists the irreducible distance regular $(2n+1)$-coloring.*

Proof. If for some $i,\ 2 \le i \le k-2$, the colors i and $i+1$ have the same degree triples then the degree triples are coincide for all colors from 2 to $k-1$ and the coloring is reducible. Otherwise all degree triples are different and the coloring is not reducible. Using Lemma 6 we get $k \le 2n+1$.

Now we are constructing the $(2n+1)$-coloring φ of $\mathbf{Z}^n$ using the coloring ψ of the $2n$-dimensional Hamming space $\mathbf{F}^{2n} = \{0, 1\}^{2n}$. Define the coloring φ from ψ using Gray transform:

$$\varphi(x_1, \ldots, x_n) = \psi\left(g(x_1 \mod 4), \ldots, g(x_n \mod 4)\right), \quad x_1, \ldots x_n \in \mathbf{Z},$$

where $g(0) = 00,\ g(1) = 01,\ g(2) = 11,\ g(3) = 10$. If ψ is the perfect coloring of $\mathbf{F}^{2n}$ then φ is also perfect with the same parameter matrix.

Let $\psi : \mathbf{F}^{2n} \longrightarrow \{1, \ldots, 2n+1\}$ be the distance coloring with respect to the all-zero vertex, i.e. $\psi(\mathbf{x}) = wt(\mathbf{x}) + 1$ for any $\mathbf{x} \in \mathbf{F}^{2n}$. It is distance regular with the parameters $d_i = i - 1,\ \ u_i = 2n - i + 1,\ \ k_i = 0,\ \ i = 1, 2, \ldots, 2n+1$. Then φ is also distance regular with the same parameters. □

6 The parameters of a completely regular code

Rewrite Theorem 2 in terms of completely regular codes:

Corollary 4. *The covering radius of an arbitrary completely regular code in the n-dimensional rectangular grid does not exceed $2n + 1$.*

We also bound the minimal distance of the first color of distance regular coloring, i.e. of the completely regular code in n-dimensional rectangular grid.

Theorem 3. *The minimal distance of an arbitrary completely regular code in n-dimensional rectangular grid is no more than 4.*

Proof. Suppose the minimal distance of the completely regular code is more than 4. Consider the distance regular coloring corresponding to the code. The spheres of radius 2 centered in code vertices do not intersect. It means that every vertex of color 3 has adjacent vertices of color 2 only into the same sphere. Take a two-dimensional subspace involving a code vertex, it has the color 1.

		3				
	3	2	**3**			
3	2	1	2	**3**		
	3	2	3			
		3				

Then the selected vertices of color 3 have different downward degrees. □

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