Thirteenth International Workshop on Algebraic and Combinatorial Coding Theory
June 15-21, 2012, Pomorie, Bulgaria *pp. 335–340*

An isomorphism between two arithmetic Fuchsian groups using different edge-pairings[1]

Cintya Wink de Oliveira Benedito cintyawink@gmail.com
Department of Telematics - University of Campinas, Brazil
Reginaldo Palazzo Junior palazzo@dt.fee.unicamp.br
Department of Telematics - University of Campinas, Brazil

Abstract. In this paper we obtain arithmetic Fuchsian groups derived from the quaternion orders associated with the tessellation $\{p, q\}$ by using different edge-pairings and we show they are isomorphic. The choice of edge-pairings is directly connected with the minimum distance of the code associated and therefore to its capacity to correct errors.

1 Introduction

The concept of geometrically uniform codes (introduced by Forney) is strongly dependent on the existence of regular tessellations in homogeneous space. The error probability, associated with a communication system when considering geometrically uniform signal constellations in the hyperbolic plane depends on the genus of a surface (surface viewed as a regular hyperbolic polygon) and the best performance is achieved with genus ≥ 2. Hence, one of our objectives is to find the group which acts transitively on the regular hyperbolic polygon (Fuchsian groups) in order to generate the signal constellation. In this direction, the signal design problem in the hyperbolic plane is to consider lattices (order of a quaternion algebra) as an algebraic structure from which the construction of signal constellations (quotient of an order by an ideal) may be realized. When considering the tessellation $\{4g, 4g\}$, with $g \geq 2$, we have shown the arithmetic Fuchsian groups associated with the diametrically opposed edge-pairings and the normal form are isomorphic. In topological quantum coding, diametrically opposed edge-pairings is the most desired because of the homological nontrivial path achieves the largest minimum distance possible. As a consequence, the resulting code has a greater capacity to correct errors among all the codes derived from the other edge-pairings.

This paper is organized as follows. In Section 2 we present some basic considerations on hyperbolic geometry and Fuchsian groups for the development of this paper. In Section 3 we consider two Fuchsian groups associated with the tessellation $\{4g, 4g\}$ by using different edge-pairings. In Section 4 we prove that the groups obtained in the previous section are isomorphic. Finally, in Section 5 we present some examples.

[1]This research is supported by FAPESP 2007/56052-8 and CNPq 141337/2010-9.

2 Hyperbolic geometry and Fuchsian groups

We consider two Euclidean models for the hyperbolic plane, the upper-half plane $\mathbb{H}^2 = \{z \in \mathbb{C} : Im(z) > 0\}$ and the Poincaré disc $\mathbb{D}^2 = \{z \in \mathbb{C} : |z| < 1\}$.

Let $PSL(2,\mathbb{R})$ be the set of all Möbius transformations, $T : \mathbb{C} \longrightarrow \mathbb{C}$, given by $T_A(z) = \frac{az+b}{cz+d}$, where $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$. A Fuchsian group Γ is a discrete subgroup of $PSL(2,\mathbb{R})$, that is, Γ consists of orientation preserving isometries $T : \mathbb{H}^2 \longrightarrow \mathbb{H}^2$, acting on $\mathbb{H}^2$ by homeomorphism, [1]. If we consider the isometry $f : \mathbb{H}^2 \longrightarrow \mathbb{D}^2$ given by $f(z) = \frac{zi+1}{z+i}$, then $\Gamma = f^{-1}\Gamma_p f$ is a subgroup of $PSL(2,\mathbb{R})$, where $T_p : \mathbb{D}^2 \longrightarrow \mathbb{D}^2$ and $T_p \in \Gamma_p < PSL(2,\mathbb{C})$ is such that $T_p(z) = \frac{az+b}{bz+\bar{a}}, \ \ a, b \in \mathbb{C}, \ \ |a|^2 - |b|^2 = 1$. Furthermore, $\Gamma \simeq \Gamma_p$.

Now, let $\mathcal{A} = (\alpha, \beta)_{\mathbb{K}}$ be a quaternion algebra over a field $\mathbb{K}$ with basis $\{1, i, j, k\}$ satisfying $i^2 = \alpha, \ j^2 = \beta$ and $k = ij = -ji$, where $\alpha, \beta \in \mathbb{K}/\{0\}$. Consider φ as an embedding of the algebra $\mathcal{A} = (\alpha, \beta)_{\mathbb{K}}$ in the algebra of matrices $M(2, \mathbb{K}(\sqrt{\alpha}))$ where

$$\varphi(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \varphi(i) = \begin{pmatrix} \sqrt{\alpha} & 0 \\ 0 & -\sqrt{\alpha} \end{pmatrix}, \varphi(j) = \begin{pmatrix} 0 & 1 \\ \beta & 0 \end{pmatrix}, \varphi(k) = \begin{pmatrix} 0 & \sqrt{\alpha} \\ -\beta\sqrt{\alpha} & 0 \end{pmatrix}.$$

Since $\varphi(i^2) = \alpha I_2, \ \varphi(j^2) = \beta I_2$ and $\varphi(ij) = \varphi(i)\varphi(j) = -\varphi(j)\varphi(i)$, it follows that φ is an isomorphism of $\mathcal{A} = (\alpha, \beta)_{\mathbb{K}}$ in the subalgebra $M(2, \mathbb{K}(\sqrt{\alpha}))$. Each element of $\mathcal{A}$ is identified with $x \longmapsto \varphi(x) = \begin{pmatrix} x_0 + x_1\sqrt{\alpha} & x_2 + x_3\sqrt{\alpha} \\ \beta(x_2 - x_3\sqrt{\alpha}) & x_0 - x_1\sqrt{\alpha} \end{pmatrix}$. If $\alpha = t^2$ with $t \in \mathbb{K}/\{0\}$, then $\mathcal{A} \simeq M(2, \mathbb{K}(\sqrt{\alpha}))$.

Let $\mathcal{A} = (\alpha, \beta)_{\mathbb{K}}$ be a quaternion algebra and R be a ring of $\mathbb{K}$. An *order* $\mathcal{O}$ in $\mathcal{A}$ is a subring of $\mathcal{A}$ containing 1, equivalently, it is a finitely generated R-module such that $\mathcal{A} = \mathbb{K}\mathcal{O}$. Hence, considering R a ring of $\mathbb{K}$ and the algebra $\mathcal{A} = (\alpha, \beta)_{\mathbb{K}}$, with $\alpha, \beta \in R$, then $\mathcal{O} = \{\alpha_0 + \alpha_1 i + \alpha_2 j + \alpha_3 k : \alpha_0, \alpha_1, \alpha_2, \alpha_3 \in R\}$, is an order in $\mathcal{A}$ denoted by $\mathcal{O} = (\alpha, \beta)_{I_{\mathbb{K}}}$. The group derived from the quaternion algebra $\mathcal{A}$ whose order is $\mathcal{O} = (\alpha, \beta)_R$, denoted by $\Gamma[\mathcal{A}, \mathcal{O}]$, is isomorphic to $PSL(2,\mathbb{R})$, [3]. Therefore, $\Gamma[\mathcal{A}, \mathcal{O}]$ is a Fuchsian group called *arithmetic Fuchsian group.*

3 Fuchsian groups Γ_{4g}

Let $\{4g, 4g\}$ be a self-dual tessellation with $g \geq 2$ in the hyperbolic plane and P_{4g} the associated regular hyperbolic polygon. In [4] it is considered Fuchsian groups Γ_{4g} whose edge-pairing generators of P_{4g} are hyperbolic transformations, T_i, where g is the genus of the surface $\mathbb{H}^2/\Gamma_{4g}$. For the normal edge-pairings, we consider the edges of P_{4g} are ordered as follows $u_1, u_2, u_1^{'}, u_2^{'}, \ldots, u_{2g-1}^{'}, u_{2g}^{'}$, such that $T_i(u_i) = u_i^{'}, \ i = 1, \ldots, 2g$. If $T_1 \in \Gamma_{4g}$ is such that $T_1(u_1) = u_1^{'}$, then, [4],

$$A_1 = \begin{pmatrix} a & \bar{b} \\ b & \bar{a} \end{pmatrix}, \text{ with } arg(a) = \frac{(g-1)\pi}{2g}, \ |a| = \tan\frac{(2g-1)\pi}{4g}$$

$$\text{and } arg(b) = \frac{-(2g+1)\pi}{4g},\ |b| = \left(\left(\tan\frac{(2g-1)\pi}{4g}\right)^2 - 1\right)^{\frac{1}{2}}. \tag{1}$$

The remaining generators are obtained by conjugations of the form

$$\begin{cases} A_i = C^{4j+1}A_1C^{-(4j+1)}, & \text{with } i \text{ even and } j = 0,\dots,g-1; \\ A_i = C^{4k}A_1C^{-4k}, & \text{with } i \text{ odd and } k = 1,\dots,g-1. \end{cases}, \tag{2}$$

where the A_i's are the transformation matrices associated with the generators T_i's of Γ_{4g}, with $i = 1,\dots,2g$ and $C = \begin{pmatrix} e^{\frac{i\pi}{4g}} & 0 \\ 0 & e^{-\frac{i\pi}{4g}} \end{pmatrix}$ is the matrix corresponding to the elliptic transformation with order $4g$.

The process of identifying Fuchsian groups derived from a quaternion algebra over a totally real algebraic number field are given by the following results.

Theorem 1. *[3] For each $g = 2^n, 3\cdot 2^n$ and $5\cdot 2^n$, where $n\in\mathbb{N}$, the elements of a Fuchsian group Γ_{4g} are identified, via isomorphism, with the elements of an order $\mathcal{O} = (\theta,-1)_R$, where $R = \left\{\frac{\delta}{2^m} : \delta\in\mathbb{Z}[\theta]\ e\ m\in\mathbb{N}\right\}$ and*

$$\theta = \begin{cases} \sqrt{2+\sqrt{2+\ldots+\sqrt{2}}} & \text{for } g = 2^n; \\ \sqrt{2+\sqrt{2+\ldots+\sqrt{2+\sqrt{3}}}} & \text{for } g = 3\cdot 2^n; \\ \sqrt{2+\sqrt{2+\ldots+\sqrt{2+\frac{\sqrt{10+2\sqrt{5}}}{2}}}} & \text{for } g = 5\cdot 2^n. \end{cases} \tag{3}$$

Theorem 2. *[3] For each $g = 2^n, 3\cdot 2^n$ and $5\cdot 2^n$, with $n\in\mathbb{N}$, the Fuchsian group Γ_{4g}, associated with the hyperbolic polygon P_{4g}, is derived from a quaternion algebra $\mathcal{A} = (\theta,-1)_{\mathbb{K}}$, over the number field $\mathbb{K} = \mathbb{Q}(\theta)$, where $[\mathbb{K}:\mathbb{Q}] = 2^n, 2^{n+1}$ and 2^{n+2}, respectively, and θ is as in* (3)*.*

Now, we will consider the Fuchsian group, denoted by Γ_{4g}^*, associated with the tessellation $\{4g,4g\}$ by using the diametrically opposed edge-pairings.
Hence, we consider the edges of P_{4g} are ordered as follows $u_1,\dots,u_{4g}$, such that $T_i^*(u_i) = u_{i+2g},\ \ i = 1,\dots,2g$. In the same way, if we have the transformation $T_1^*\in\Gamma_{4g}^*$ and so the corresponding matrix A_1^*, the remaining generators are obtained by conjugations of the form

$$A_i^* = C^{i-1}A_1^*C^{-(i-1)},\ i = 2,\dots,2g. \tag{4}$$

As $T_1^*(u_1) = u_{1+2g}$ the form of the matrix A_1^* may be obtained by the following result:

Theorem 3. *[5] Let P_p be a hyperbolic regular polygon with p edges and Γ_p the Fuchsian group associated with the tessellation $\{p,q\}$. If $T_1\in\Gamma_p$ is such* *that* $T_1(u_1) = u_{1+\frac{p}{2}}$ *then the matrix* A_1 *associated with the transformation* T_1 *is given by*

$$A_1 = \begin{pmatrix} \frac{2\cos\frac{\pi}{q}}{2\sin\frac{\pi}{p}} & \frac{\sqrt{2\cos\frac{\pi}{p}+2\cos\frac{\pi}{q}}\cdot e^{i\left(\frac{p+1}{p}\right)\pi}}{2\sin\frac{\pi}{p}} \\ \frac{\sqrt{2\cos\frac{\pi}{p}+2\cos\frac{\pi}{q}}\cdot e^{-i\left(\frac{p+1}{p}\right)\pi}}{2\sin\frac{\pi}{p}} & \frac{2\cos\frac{\pi}{q}}{2\sin\frac{\pi}{p}} \end{pmatrix}. \quad (5)$$

Using an appropriate set of edge-pairings of P_p*, all the side pairing transformations are obtained by conjugation of the form* $T_i = T_{C^{r_i}} \circ T_1 \circ T_{C^{-r_i}}$*, where* $T_{C^{r_i}}$ *is a power of matrix* C*.*

Remark 1. *We can prove that Theorems* 1 *and* 2*, [3], are also valid if we consider the group* Γ^*_{4g} *but due to space limitation we omit the proofs. In this way, we can find the arithmetic Fuchsian group* Γ^*_{4g} *derived from a quaternion algebra using the diametrically opposed edge-pairings.*

4 Isomorphism between arithmetic Fuchsian groups

The purpose of this section is to show that there is an isomorphism between arithmetic Fuchsian groups derived from each fixed tessellation $\{p, q\}$. We prove this for the groups Γ_{4g} and Γ^*_{4g}.

Lemma 1. *[5] Let* $\Gamma \subset PSL(2,\mathbb{R})$ *be a finitely generated Fuchsian group with generators* $G_1, \ldots G_l$*. Then*

$$G_i = \frac{1}{2^s}\begin{pmatrix} x_i + y_i\sqrt{\theta} & z_i + w_i\sqrt{\theta} \\ -z_i + w_i\sqrt{\theta} & x_i - y_i\sqrt{\theta} \end{pmatrix}, \quad (6)$$

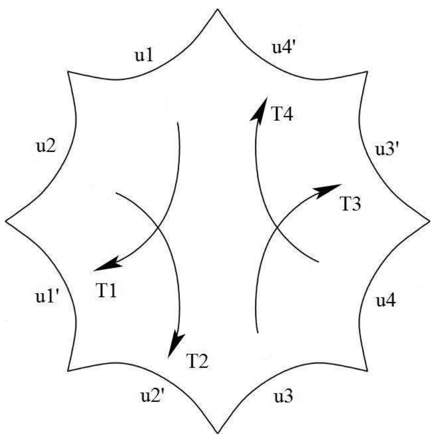
where $G_i \in M(2, \mathbb{K}(\sqrt{\theta}))$*,* $s \in \mathbb{N}$*,* $\theta, x_i, y_i, z_i, w_i \in \mathbb{K}$*, with* $i = 1, \ldots, l$ *and* $\mathbb{K}$ *a totally real algebraic number field. Furthermore, any element* $T \in \Gamma$ *has the same form as that of the generators of* Γ*.*

Theorem 4. *Let* $\Gamma_{4g}, \Gamma^*_{4g} \subset PSL(2,\mathbb{C})$ *be arithmetic Fuchsian groups associated with the tessellation* $\{4g, 4g\}$ *using the normal edge-paring and the diametrically opposed edge-pairings, respectively. Then* $\Gamma_{4g} \simeq \Gamma^*_{4g}$*.*

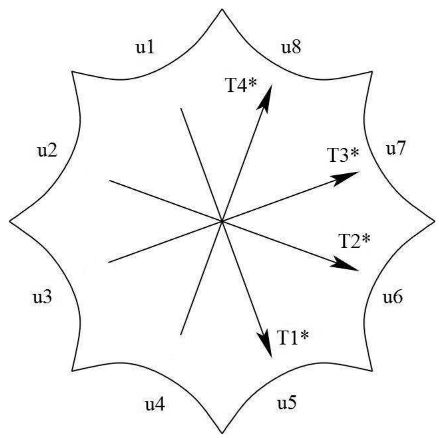
Proof. Fixed a genus g, we consider Γ_1 and Γ_2 as Fuchsian groups in $\mathbb{H}^2$, that is, $PSL(2,\mathbb{R})$ subgroups. So, $\Gamma_1 = f^{-1}\Gamma_{4g}f$ and $\Gamma_2 = f^{-1}\Gamma^*_{4g}f$, where $f : z \longmapsto \frac{zi+1}{z+i}$. Furthermore, if we consider the isomorphism of groups $\phi_1 : \Gamma_1 \longrightarrow \Gamma_{4g}$ and $\phi_2 : \Gamma_2 \longrightarrow \Gamma^*_{4g}$ given by $\phi_1(T) = \phi_2(T) = f^{-1}Tf$, we have that $\Gamma_1 \simeq \Gamma_{4g}$ and $\Gamma_2 \simeq \Gamma^*_{4g}$. For each $i = 1, \ldots, 2g$, let A_i and A^*_i be the corresponding transformation matrices T_i and T^*_i, respectively. Then, for $i = 1, \ldots, 2g$ we have that $G_i = f^{-1}A_if$ and $G^*_i = f^{-1}A^*_if$, are the generators of the groups Γ_1 and Γ_2, respectively. As $\Gamma_1, \Gamma_2 \subset PSL(2,\mathbb{R})$, by Lemma 1 it follows that both generators G_i and G_i^* are of the form given in (6). Hence, by the construction of Γ_1 and Γ_2 from Γ_{4g} and Γ_{4g}^*, we have $G_i, G_i^* \subset M(2, \mathbb{Q}(\sqrt{\theta}))$, both for the same value of θ, where θ depends on the genus g and is as in (3). In this way, there exists an isomorphism $\psi : \Gamma_1 \longrightarrow \Gamma_2$ in which $\psi(G_i) = G_i^*$. Therefore, by the chain of isomorphisms $\Gamma_{4g} \simeq \Gamma_1 \simeq \Gamma_2 \simeq \Gamma_{4g}^*$, it follows that $\Gamma_{4g} \simeq \Gamma_{4g}^*$. □

5 Examples

As an application of the previously established concepts, we present next two examples where we derive the generators of the Fuchsian groups Γ_{4g} and Γ_{4g}^*, for $g = 2$. The normal and diametrically opposed edge-pairings constructed previously are shown in the following figures.



P_8-normal edge-pairings



P_8-diametrically opposed edge-pairings

Example 1. *Let P_8 be the regular hyperbolic polygon associated with the tessellation $\{8, 8\}$. Let us consider the normal form for the edge-pairings of P_8. Using the equalities in* (1) *and* (2)*, and the fact that $G_i = f^{-1}A_i f$, with $i = 1, \cdots, 4$ we obtain the following generators of the arithmetic Fuchsian group Γ_8:*

$$G_1 = \begin{pmatrix} \frac{x_1 - x_1\sqrt[4]{2}}{2} & \frac{x_1 - y_1\sqrt[4]{2}}{2} \\ \frac{-x_1 - y_1\sqrt[4]{2}}{2} & \frac{x_1 + x_1\sqrt[4]{2}}{2} \end{pmatrix} = \varphi\left(\tfrac{x_1}{2} - \tfrac{x_1}{2}i + \tfrac{x_1}{2}j - \tfrac{y_1}{2}k\right), \quad G_2 = \begin{pmatrix} \frac{x_1 + x_1\sqrt[4]{2}}{2} & \frac{x_1 + y_1\sqrt[4]{2}}{2} \\ \frac{-x_1 + y_1\sqrt[4]{2}}{2} & \frac{x_1 - x_1\sqrt[4]{2}}{2} \end{pmatrix} = \varphi\left(\tfrac{x_1}{2} + \tfrac{x_1}{2}i + \tfrac{x_1}{2}j + \tfrac{y_1}{2}k\right),$$

$$G_3 = \begin{pmatrix} \frac{x_1 - x_1\sqrt[4]{2}}{2} & \frac{-x_1 + y_1\sqrt[4]{2}}{2} \\ \frac{x_1 + y_1\sqrt[4]{2}}{2} & \frac{x_1 + x_1\sqrt[4]{2}}{2} \end{pmatrix} = \varphi\left(\tfrac{x_1}{2} - \tfrac{x_1}{2}i - \tfrac{x_1}{2}j + \tfrac{y_1}{2}k\right), \quad G_4 = \begin{pmatrix} \frac{x_1 + x_1\sqrt[4]{2}}{2} & \frac{-x_1 - y_1\sqrt[4]{2}}{2} \\ \frac{x_1 - y_1\sqrt[4]{2}}{2} & \frac{x_1 - x_1\sqrt[4]{2}}{2} \end{pmatrix} = \varphi\left(\tfrac{x_1}{2} + \tfrac{x_1}{2}i - \tfrac{x_1}{2}j - \tfrac{y_1}{2}k\right),$$

where $x_1 = 2 + \sqrt{2}$ and $y_1 = \sqrt{2}$. Hence, according to Theorem 1, the quaternion order associated with the Fuchsian group Γ_8 is $\mathcal{O} = (\sqrt{2}, -1)_R$, where $R = \{\frac{\delta}{2^m}$:

$\delta \in \mathbb{Z}[\sqrt{2}]$ and $m \in \mathbb{N}\}$, and according to Theorem 2, Γ_8 is derived from the quaternion algebra $\mathcal{A} = (\sqrt{2}, -1)_{\mathbb{K}}$, with $\mathbb{K} = \mathbb{Q}(\sqrt{2})$ and $[\mathbb{K} : \mathbb{Q}] = 2$.

Example 2. *Let P_8 be the regular hyperbolic polygon associated with the tessellation $\{8, 8\}$. Let us consider the diametrically opposed edge-pairings of P_8. Using the equalities in (4), Theorem 3 and the fact that $G_i^* = f^{-1}A_i f$, $i = 1, \cdots, 4$ we obtain the following generators of the arithmetic Fuchsian group Γ_8^*:*

$$G_1^* = \begin{pmatrix} \frac{x_1+y_1\sqrt[4]{2}}{2} & \frac{-w_1\sqrt[4]{2}}{2} \\ \frac{-w_1\sqrt[4]{2}}{2} & \frac{x_1-y_1\sqrt[4]{2}}{2} \end{pmatrix} = \varphi(\tfrac{x_1}{2} + \tfrac{y_1}{2}i - \tfrac{w_1}{2}k), \quad G_2^* = \begin{pmatrix} \frac{x_1-w_1\sqrt[4]{2}}{2} & \frac{y_1\sqrt[4]{2}}{2} \\ \frac{y_1\sqrt[4]{2}}{2} & \frac{x_1+w_1\sqrt[4]{2}}{2} \end{pmatrix} = \varphi(\tfrac{x_1}{2} - \tfrac{w_1}{2}i + \tfrac{y_1}{2}k),$$

$$G_3^* = \begin{pmatrix} \frac{x_1-w_1\sqrt[4]{2}}{2} & \frac{-y_1\sqrt[4]{2}}{2} \\ \frac{-y_1\sqrt[4]{2}}{2} & \frac{x_1+w_1\sqrt[4]{2}}{2} \end{pmatrix} = \varphi(\tfrac{x_1}{2} - \tfrac{w_1}{2}i - \tfrac{y_1}{2}k), \quad G_4^* = \begin{pmatrix} \frac{x_1-y_1\sqrt[4]{2}}{2} & \frac{-w_1\sqrt[4]{2}}{2} \\ \frac{-w_1\sqrt[4]{2}}{2} & \frac{x_1+y_1\sqrt[4]{2}}{2} \end{pmatrix} = \varphi(\tfrac{x_1}{2} - \tfrac{y_1}{2}i - \tfrac{w_1}{2}k),$$

where $x_1 = 2 + 2\sqrt{2}, y_1 = \sqrt{2}$ and $w_1 = 2 + \sqrt{2}$. Hence, according to Remark 1 the quaternion order associated with the Fuchsian group Γ_8^* derived from the quaternion algebra $\mathcal{A} = (\sqrt{2}, -1)_{\mathbb{K}}$, is $\mathcal{O} = (\sqrt{2}, -1)_R$, where $R = \{\frac{\delta}{2^m} : \delta \in \mathbb{Z}[\sqrt{2}]$ and $m \in \mathbb{N}\}$, $\mathbb{K} = \mathbb{Q}(\sqrt{2})$ and $[\mathbb{K} : \mathbb{Q}] = 2$.

Remark 2. *Although the generators have the same form, as given in (6), they are distinct. This implies that different Fuchsian groups are being generated, however they are isomorphic. Furthermore, both set of generators yield arithmetic Fuchsian groups derived from the same quaternion algebra $\mathcal{A} = (\sqrt{2}, -1)_{\mathbb{K}}$.*

Remark 3. *These results extend to other tessellations other than the self-dual tessellation $\{4g, 4g\}$.*

References

[1] S. Katok, *Fuchsian Groups,* The University of Chicago Press, Chicago, 1992.

[2] G. D. Forney, Jr., *Geometrically uniform codes,* IEEE Trans. Inform. Theory, vol. IT 37, 1241-1260, 1991.

[3] V. L. Vieira, R. Palazzo,Jr. and M. B. Faria, *On the arithmetic Fuchsian groups derived from quaternion orders,* IEEE Trans. Inform. Theory, Intern. Telecom. Symp., 586-591, 2006.

[4] E. D. Carvalho, *Identification of lattices from genus of compact surface,* IEEE Trans. Inform. Theory, Intern. Telecom. Symp. 146-151, 2006.

[5] V. L. Vieira, *Arithmetic Fuchsian groups identified in quaternion orders for signal constellations construction,* Doctoral Dissertation, FEEC-UNICAMP, 2007 (in Portuguese).