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Classification of the odd sets in PG$(4,4)$ [1]

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Abstract. A set $\mathscr{K}$ in PG$(d,4)$, $d \geq 2$, is *odd if every line meets* $\mathscr{K}$ *in an odd number of points. We show there are exactly* 45 *inequivalent odd sets in PG*$(4,4)$*, which are classified to three cycles by disflection.*

1 Introduction

Tallini Scafati [6] posed the problem to characterize all sets in PG(d,q) with only 1-, n- and $(q+1)$-secants. Such sets with $n=3$ in PG$(d,4)$ are called *odd sets*. The odd sets appear in coding theory to study the extendability of quaternary linear codes with even weights whose minimum distance is congruent to 2 modulo 4, see [7]. We denote by O_d the set of odd sets in PG$(d,4)$. Hirschfeld and Hubaut [3] classified O_d for $d \leq 3$ up to projective equivalence, thus leaving the cases with $d \geq 4$ still unclassified. Brian Sherman [5] introduced *disflection* and characterized the odd sets as varieties (Theorem 1), see [1]. We shall classify O_4 up to projective equivalence by way of Sherman's method.

2 Main results

In this section, we give some preliminaries and main results. Let Π_r be an r-flat PG(d,q). Take Π_r and Π_s in PG(d,q) such that $\Pi_r \cap \Pi_s = \emptyset$. For a set $\mathscr{K}$ in Π_r, $\Pi_s\mathscr{K} = \bigcup_{P\in\Pi_s, Q\in\mathscr{K}} \langle P, Q\rangle$ is called a *cone* with *vertex* Π_s and *base* $\mathscr{K}$, where $\langle P, Q\rangle$ stands for the line through P and Q.

For $\mathscr{K} \in O_d$ and a hyperplane π of Π_d, define the map $\delta_\pi : O_d \to O_d$ by $\mathscr{K}\delta_\pi = \mathscr{K}\nabla\pi$, with $\mathscr{K}\nabla\pi = (\mathscr{K}^c \cap \pi^c) \cup (\mathscr{K} \cap \pi)$, where $\mathscr{K}^c = \Pi_d \setminus \mathscr{K}$. The map δ_π is called a *disflection by* π. Note that $|\mathscr{K}\nabla\mathscr{K}'| = |\Pi_d| - |\mathscr{K}| - |\mathscr{K}'| + 2|\mathscr{K} \cap \mathscr{K}'|$ for $\mathscr{K}$, $\mathscr{K}' \in O_d$.

Theorem 1 ([1]). *Every odd set in* PG$(d,4)$ *is uniquely expressed as* $\mathbf{V}(E^2 + E + H)$*, where* $E = \sum_{0<i,j,k<d} c_{ijk}x_ix_jx_k$ *and* H *is Hermitian.*

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We found a new odd set $\mathscr{V} = \mathbf{V}(E^2 + E)$ with $E = x_0x_3x_4 + x_1x_2x_4$, which cannot be obtained by disflections from any cone with base from O_d with $d \le 3$.

For $X = \mathbf{P}(x_0, x_1, x_2, x_3, x_4)$ and $P = \mathbf{P}(p_0, p_1, p_2, p_3, 1) \in \mathrm{PG}(4, 4)$, let $h(X) = x_0x_3 + x_1x_2$ and $g_P(X) = p_3x_0 + p_2x_1 + p_1x_2 + p_0x_3$.

Lemma 1. *For $R_i, R_j \in \mathrm{PG}(4, 4)$ and $\lambda_i, \lambda_j \in \mathbb{F}_4$, the following equalities hold:*

(1) $g_P(\sum_i \lambda_i R_i) = \sum_i \lambda_i g_P(R_i)$.

(2) $h(\sum_i \lambda_i R_i) = \sum_i \lambda_i^2 h(R_i) + \sum_{i<j} \lambda_i\lambda_j g_{R_i}(R_j)$.

Let $\mathscr{P}_4 = \mathbf{V}(h(X) + x_4^2)$ and $N\mathscr{H}_3 = \mathbf{V}(h(X))$, where $N = \mathbf{P}(0, 0, 0, 0, 1)$, and $\Pi = \mathbf{V}(x_4)$. Then we have $\mathscr{V} = \mathscr{P}_4 \cup N\mathscr{H}_3 \cup \Pi$. A line meeting $\mathscr{V}$ in i points is called an *i-line*.

Lemma 2. *Let $P = \mathbf{P}(p_0, p_1, p_2, p_3, 1) \in \mathscr{V} \setminus \Pi$. Then,*

(1) *There are 25 5-lines on P. Let $\mathscr{C}_P$ be the set of points in the 25 5-lines.*

(2) *It holds that $\Pi \cap \mathscr{C}_P = \mathbf{V}(x_4) \cap \mathbf{V}(h(X) + g_P^2(X))$ with $g_P(X) = p_3x_0 + p_2x_1 + p_1x_2 + p_0x_3$. Moreover, $\mathscr{C}_P = \mathbf{V}(h(X) + g_P^2(X) + g_P(X)x_4)$ for $P \in N\mathscr{H}_3$, and $\mathscr{C}_P = \mathbf{V}(h(X) + g_P^2(X) + g_P(X)x_4 + x_4^2)$ for $P \in \mathscr{P}_4$.*

(3) *Let $\mathscr{C}_P = \mathbf{V}(h_P(X))$ and $\mathscr{V}_P = \mathbf{V}(h_P(X) + x_4^2) \cup \mathbf{V}(h_P(X)) \cup \mathbf{V}(x_4)$. Then $\mathscr{V}_P = \mathscr{V}$.*

(4) *$\mathscr{P}_P = \mathbf{V}(h_P(X) + x_4^2)$ is a parabolic quadric whose nucleus is P.*

We express $\mathscr{K} \sim \mathscr{K}'$ if $\mathscr{K}$ and $\mathscr{K}'$ are projectively equivalent. Let $N(\mathscr{K}) := |\{\mathscr{K}' \in O_d \mid \mathscr{K} \sim \mathscr{K}'\}|$.

Lemma 3 ([1])**.** *The dimension of O_d as a binary vector space is $(d^3 + 3d^2 + 5d + 3)/3$.*

It follows that the total of all $N(\mathscr{K})$ for the inequivalent odd sets $\mathscr{K}$ in O_4 is equal to $2^{\dim(O_4)} = 2^{45}$. An odd set $\mathscr{K}$ is *non-singular* if every point of $\mathscr{K}$ is on some 3-line.

Lemma 4. *Take Π_{d-s-1} and Π_s in $\mathrm{PG}(d, 4)$ so that $\Pi_{d-s-1} \cap \Pi_s = \emptyset$. For a non-singular odd set $\mathscr{K}$ in Π_{d-s-1}, it holds that*

$$N(\Pi_s\mathscr{K}) = N(\mathscr{K}) \times \frac{\theta_d\theta_{d-s-1}}{\theta_s}.$$

Let $\mathcal{F}_{d-1}$ be the set of hyperplanes of $\mathrm{PG}(d,4)$.

Lemma 5. *For an odd set* $\mathscr{K}$ *in* Π_d *and a hyperplane* Δ *of* Π_d*, let* $\mathscr{K}' = \mathscr{K}\nabla\Delta$*,* $s = |\{\pi \in \mathcal{F}_{d-1} \mid \mathscr{K}\nabla\pi \sim \mathscr{K}'\}|$*, and* $s' = |\{\pi' \in \mathcal{F}_{d-1} \mid \mathscr{K}'\nabla\pi' \sim \mathscr{K}\}|$*. Then*

$$N(\mathscr{K}') = N(\mathscr{K}) \times \frac{s}{s'}.$$

Lemma 6. $N(\mathscr{V}) = 263983104$.

Theorem 2. *There are exactly* 45 *inequivalent odd sets in* $\mathrm{PG}(4,4)$ *and they are classified to three cycles by disflection.*

See Tables 1-3 for the above result.

Note 1. *In the Tables, we refer to* [1], [2], [4] *for terminologies. For example,* $\mathscr{U}_n$ *is a non-singular Hermitian variety in* $\mathrm{PG}(n,4)$ [4]*. We also note that* $disf_r^s\mathscr{K}$ *is an odd set given by disflecting* $\mathscr{K}$ *at least s times.*

References

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Table 1: Odd sets in PG(4, 4) with disflections by solids

	$\mathscr{K}$	$\lvert\mathscr{K}\rvert$	Π_3	Π_2	$\Pi_1\mathscr{U}_1$	$\Pi_0\mathscr{U}_2$	$\mathscr{U}_3$	$\Pi_0\mathscr{F}_a$	$\Pi_0\mathscr{P}$	$\Pi_0\mathscr{O}^c$	$\mathscr{R}_3$	$\mathscr{K}^*$	$\mathscr{S}_{IV}$	$\mathscr{S}_{III}$	$\mathscr{S}_{II}$	$\mathscr{T}$
1	Π_4	341	2													
2	Π_3	85	1	3												
3	$\Pi_2\mathscr{U}_1$	213	3	2	4											
4	$\Pi_1\mathscr{U}_2$	149		4	3	5										
5	$\Pi_0\mathscr{U}_3$	181			5	4	6									
6	$\mathscr{U}_4$	165				6	5									
7	$\Pi_0\mathscr{P}_{II}$	117		8	9			14								
8	$\Pi_0\mathscr{P}_{III}$	181	9	7	8				13							
9	$\Pi_0\mathscr{P}_{IV}$	245	8		7					12						
10	$\Pi_0\mathscr{R}_3$	213	10		11				12	13	16					
11	$\Pi_0\mathscr{K}^*$	149		11	10	15		13	14			17				
12	$\Pi_0\mathscr{S}_{IV}$	133		13		14		15	10	9			18			
13	$\Pi_0\mathscr{S}_{III}$	165		12	14	13		11	8,15	10				19		
14	$\Pi_0\mathscr{S}_{II}$	197			13	12		7	11	15					20	
15	$\Pi_0\mathscr{T}$	181			15	11		12	13	14						21
16	$disf\Pi_0\mathscr{R}_3$	149		16		17					9	22	18	18		
17	$disf\Pi_0\mathscr{K}^*$	181			17	16	21				22	10		20	19	23
18	$disf\Pi_0\mathscr{S}_{IV}$	189			19		20			18	19		20	16	21	20
19	$disf\Pi_0\mathscr{S}_{III}$	173			14	20	19		19		18	20	16	12,21	17	19
20	$disf\Pi_0\mathscr{S}_{II}$	157				19	18	20				19	21	17	13	18
21	$disf\Pi_0\mathscr{T}$	165				21	17					23	20	19	18	14,22
22	$disf_1^2\Pi_0\mathscr{T}$	181			22		23				17	16				21
23	$disf_2^2\Pi_0\mathscr{T}$	165				23	22					21				17

Table 2: Odd sets in PG(4, 4) with disflections by solids (continued)

	$\mathscr{K}$	$\lvert\mathscr{K}\rvert$	Π_3	Π_2	$\Pi_1\mathscr{U}_1$	$\Pi_0\mathscr{U}_2$	$\mathscr{U}_3$	$\Pi_0\mathscr{F}_a$	$\Pi_0\mathscr{P}$	$\Pi_0\mathscr{O}^c$	$\mathscr{R}_3$	$\mathscr{K}^*$	$\mathscr{S}_{IV}$	$\mathscr{S}_{III}$	$\mathscr{S}_{II}$	$\mathscr{T}$
24	$\mathscr{V}$	221	25							26	27					
25	$disf_1\mathscr{V}$	205	24						27		26					
26	$disf_2\mathscr{V}$	157		27					30	24	25	29	28	32		
27	$disf_3\mathscr{V}$	141		26				29	25		24	30	32	31		
28	$disf_1^2\mathscr{V}$	165			31			34	32		31	35	26	37		33
29	$disf_2^2\mathscr{V}$	173			30			27	29		30	26,36		35	33	37
30	$disf_3^2\mathscr{V}$	189			29				26	30	29,37	27		34	35	36
31	$disf_4^2\mathscr{V}$	197			28					32,33	28,35			27	36	34
32	$disf_5^2\mathscr{V}$	181			32				28,35	31	32,33	34	27	26,36	37	35
33	$disf_1^3\mathscr{V}$	181				34			35	31	32	34	38	36,40	29,39,42	28,44
34	$disf_2^3\mathscr{V}$	149				33		28,35				32,33,45	37,39,42	30,41	43	31
35	$disf_3^3\mathscr{V}$	165				35		34	32,33		31	28,35,44	36,40	29,37,39,42	41,45	32,45·
36	$disf_4^3\mathscr{V}$	157				37		36				29,37,42	35,44	32,33,45	31	30,41
37	$disf_5^3\mathscr{V}$	173				36			37		30	36,40	34	28,35,44	32,45	29,42
38	$disf_1^4\mathscr{V}$	141					43	39				41	33			43
39	$disf_2^4\mathscr{V}$	173					39	38	39	43	41	40	34	35	33	39,42
40	$disf_3^4\mathscr{V}$	157					41	40	41		43	37,39,42	35	33,45		41
41	$disf_4^4\mathscr{V}$	189					40		40	41	39,42	38		34	33,44	40,36
42	$disf_5^4\mathscr{V}$	173					42		42		41	36,40	34	35,44	33,45	37,39,42
43	$disf_6^4\mathscr{V}$	205					38			39	40				34	38
44	$disf_7^4\mathscr{V}$	165					45					35,44	36	37,42	41	33,45
45	$disf_8^4\mathscr{V}$	181					44				45	34		36,40	37,42	35,44

Table 3: *Types* and numbers of odd sets in O_4

$\mathscr{K}$	$\lvert\mathscr{K}\rvert$	$N(\mathscr{K})$	$\mathscr{K}$	$\lvert\mathscr{K}\rvert$	$N(\mathscr{K})$
Π_4	341	1	$\mathscr{V}$	221	263983104
Π_3	85	341	$disf_1\mathscr{V}$	205	263983104
$\Pi_2\mathscr{U}_1$	213	57970	$disf_2\mathscr{V}$	157	22438563840
$\Pi_1\mathscr{U}_2$	149	1623160	$disf_3\mathscr{V}$	141	22438563840
$\Pi_0\mathscr{U}_3$	181	12985280	$disf_1^2\mathscr{V}$	165	179508510720
$\mathscr{U}_4$	165	18887680	$disf_2^2\mathscr{V}$	173	224385638400
$\Pi_0\mathscr{P}_{II}$	117	2086920	$disf_3^2\mathscr{V}$	189	224385638400
$\Pi_0\mathscr{P}_{III}$	181	5843376	$disf_4^2\mathscr{V}$	197	179508510720
$\Pi_0\mathscr{P}_{IV}$	245	973896	$disf_5^2\mathscr{V}$	181	1077051064320
$\Pi_0\mathscr{R}_3$	213	175301280	$disf_1^3\mathscr{V}$	181	897542553600
$\Pi_0\mathscr{K}^*$	149	876506400	$disf_2^3\mathscr{V}$	149	897542553600
$\Pi_0\mathscr{S}_{IV}$	133	311646720	$disf_3^3\mathscr{V}$	165	5385255321600
$\Pi_0\mathscr{S}_{III}$	165	1869880320	$disf_4^3\mathscr{V}$	157	2692627660800
$\Pi_0\mathscr{S}_{II}$	197	667814400	$disf_5^3\mathscr{V}$	173	2692627660800
$\Pi_0\mathscr{T}$	181	1753012800	$disf_1^4\mathscr{V}$	141	22438563840
$disf\Pi_0\mathscr{R}_3$	149	11219281920	$disf_2^4\mathscr{V}$	173	673156915200
$disf\Pi_0\mathscr{K}^*$	181	56096409600	$disf_3^4\mathscr{V}$	157	1346313830400
$disf\Pi_0\mathscr{S}_{IV}$	189	39890780160	$disf_4^4\mathscr{V}$	189	1346313830400
$disf\Pi_0\mathscr{S}_{III}$	173	239344680960	$disf_5^4\mathscr{V}$	173	8077882982400
$disf\Pi_0\mathscr{S}_{II}$	157	85480243200	$disf_6^4\mathscr{V}$	205	22438563840
$disf\Pi_0\mathscr{T}$	165	112192819200	$disf_7^4\mathscr{V}$	165	4308204257280
$disf_1^2\Pi_0\mathscr{T}_1$	181	16828922880	$disf_8^4\mathscr{V}$	181	4308204257280
$disf_2^2\Pi_0\mathscr{T}_2$	165	16828922880			
			Total		2^{45}