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On the types of book spreads of $PG(7,2)$

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Abstract. We construct line spreads in $PG(7,2)$ of a special kind, namely those which are *book spreads*. There are three main types of book spreads and several subtypes based on the relation of book spreads in $PG(7,2)$ to the 9 book spreads in $PG(5,2)$ which have been classified in a previous work. We present classification results for some of these subtypes.

1 Introduction

A *t-spread* in $PG(n,q)$ is a partition of the points of the projective space by t-flats (t-dimensional subspaces). Usually 1-spreads are called line spreads, or just spreads.

Spreads in projective spaces are used to construct constant dimention codes and spread codes [3], [9], [12]. Spreads have been widely studied in the last several decades and quite many constructions of spreads have been found [1], [8]. Classification results are known for spreads in $PG(3,q)$ with certain automorphisms [5], [6], [7], for maximal partial spreads in $PG(3,2)$ [13], $PG(3,3)$ [13], $PG(3,4)$ [13], [14], and $PG(4,2)$ [4], for spreads in $PG(5,2)$ [10], and for maximal partial spreads of size 45 in $PG(3,7)$ [2].

An (n,q,p,s) *book* in $PG(n,q)$ is a collection of p-flats called *pages*, which cover the whole projective space and intersect in a common s-flat called *spine*. Any point outside the spine is in exactly one page. An (n,q,p,s) *book t-spread* is a t-spread such that the points of each page of an (n,q,p,s) book and the points of the spine are partitioned by t-flats of this t-spread.

The idea about book spreads belongs to the first author. In some unpublished research in 2004 he considered spreads in $PG(5,2)$ of a special kind, namely $(5,2,3,1)$ book spreads. Using theoretical considerations he succeeded in classifying them into nine different kinds. The result was partially checked by computer by McDonough. Line spreads in $PG(5,2)$ were considered by Mateva and Topalova in [10], where, with computer assistance, 131044 inequivalent spreads were found. Among them there are only 9 spreads for which there are at least five 3-dimensional subspaces containing 5 spread lines. However it is not clear that they are all book spreads. Therefore Topalova carried through some new computer-aided work, which concentrated solely on $(5,2,3,1)$ book spreads. The results coincided with both the final, and some intermediate theoretical results, proved by Shaw [11].

Most of the spreads in $PG(5,2)$ [10] have very little symmetry and presumably their properties do not warrant further consideration. The nine book spreads are among the most interesting of them. We believe that book spreads with other parameters are also of special interest. Their lines partition not only the whole projective space, but also subspaces covering it, and therefore spreads with the richest automorphism groups and other interesting geometric properties are among them.

In the present work we consider book spreads in $PG(7,2)$ with parameters $(7,2,5,3)$. Their full classification by computer is a very hard task. We use our knowledge of book spreads in $PG(5,2)$ to classify some classes of $(7,2,5,3)$ book spreads, which are of particular interest.

2 $(7,2,5,3)$ book spreads

A spread in $PG(5,2)$ has 21 lines. The spine of a $(5,2,3,1)$ book spread is a line. There are 5 pages. Each page is a 3-flat with 5 spread lines (the spine and 4 other lines).

A spread in $PG(7,2)$ has 85 lines. The spine of a $(7,2,5,3)$ book spread is a 3-flat containing 5 spread lines. Let us denote them s_1, s_2, s_3, s_4 and s_5. There are 5 pages. Denote them P_1, P_2, P_3, P_4 and P_5. Each page is a 5-flat with 21 spread lines (the 5 lines of the spine and 16 other lines). The type of the spreads in the five pages can be used to define the following three types of $(7,2,5,3)$ book spreads, namely

- $(7,2,5,3)_1$:
  The spread lines in each page form a $(5,2,3,1)$ book spread with spine s_1

- $(7,2,5,3)_2$:
  The spread lines in page P_i form a $(5,2,3,1)$ book spread with spine s_i for $i = 1,2,3,4,5$.

- $(7,2,5,3)_3$:
  all the remaining $(7,2,5,3)$ book spreads.

We later consider only $(7,2,5,3)_1$ and $(7,2,5,3)_2$ book spreads. We then set the additional restriction that the $(5,2,3,1)$ book spreads of the 5 pages should be isomorphic. We present in Table 1 information about the 9 page types which correspond to the nine $(5,2,3,1)$ book spreads of [11]. $|Aut|$ is the order of the automorphism group, and n_i is the number of 3-flats containing i spread lines. Pages of type 1 contain the normal spread in $PG(5,2)$.

Table 1: The nine $(5, 2, 3, 1)$ book spreads (see [11]) define nine page types of $(7, 2, 5, 3)_1$ and $(7, 2, 5, 3)_2$ book spreads

page type	$\|Aut\|$	n_3	n_4	n_5
1	362880	0	0	21
2	1728	0	12	9
3	1152	0	16	5
4	108	18	6	6
5	72	24	4	5

page type	$\|Aut\|$	n_3	n_4	n_5
6	384	32	0	5
7	36	24	3	5
8	288	24	0	5
9	5760	0	0	5

3 Construction method

All computer results are obtained by our own C++ programs. We assign the spread lines numbers from 1 to 85. We construct spreads with the following properties:

- Lines $1, 2, 3, 4$, and 5 compose the spine
- Except for the spine lines, P_i contains lines $16(i-1)+a$, where $i = 1, 2, ..., 5$ and $a = 6, 7, ...21$.
- For each $n = 1, 2, ..., 4$ it holds: Lines $16(i-1)+4(n-1)+b$ $(b = 6, 7, 8, 9)$ are from one and the same page of the $(5, 2, 3, 1)$ book spread in P_i, $i = 1, 2, ..., 5$.

Without loss of generality we fix 7 lines of the spread, namely the first 6 lines and line 22. We then use backtrack search to choose the remaining spread lines from a set D of 1747 lines (out of all 10795 lines of the projective space) that are skew to each of the fixed 7 ones. We apply isomorphism check on the partial solutions after adding the lines of a whole page of the $(7, 2, 5, 3)$ book spread, namely after adding lines $21, 37, 53, 69$ and 85.

We perform the isomorphism test in the following way: Before starting the search we find all automorphisms which stabilize the fixed seven lines. Their number is 108. We also find and save for each line of the set D an automorphism mapping it to line 1, an automorphism mapping it to line 2 and fixing line 1, an automorphism mapping it to line 6 and fixing lines $1, 2, ..., 5$, and an automorphism mapping it to line 22 and fixing lines $1, 2, ..., 21$. To test the current solution for minimality, we use the above listed automorphisms to map the spread lines to the 7 fixed lines in all possible ways, and then in each case we apply the 108 automorphisms, which stabilize the fixed lines. If one of them maps the current solution to a lexicographically smaller one, we drop the current partial solution.

4 Classification results

We find out that all $(7,2,5,3)$ book spreads with all pages of type 1 are both $(7,2,5,3)_1$ and $(7,2,5,3)_2$ book spreads. Their number is 13. In Tables 2 to 6 we present classification results for $(7,2,5,3)_1$ and $(7,2,5,3)_2$ book spreads with some of the page types. In column $|Aut|$ we present the orders of the automorphism groups of the constructed spreads, and in the other column the number of spreads with the corresponding automorphism group order. As it was expected, $(7,2,5,3)_1$ and $(7,2,5,3)_2$ book spreads include many spreads with rich automorphism groups. This makes us believe that their further study is of particular interest.

Table 2: $(7,2,5,3)_1/(7,2,5,3)_2$ book spreads with all pages of type 1

$\|Aut\|$	spreads
5922201600	1
1105920	1
221184	1
207360	1
73728	1

$\|Aut\|$	spreads
18432	1
15360	1
13824	1
9216	1
6480	1

$\|Aut\|$	spreads
3456	1
1152	1
432	1
All	13

Table 3: $(7,2,5,3)_1$ book spreads with all pages of type 2

$\|Aut\|$	spreads
1	57823
2	8624
3	1344
4	655
6	2027
8	264
9	31
12	305
16	14
18	445
24	236
32	10
36	87
48	35
54	74

$\|Aut\|$	spreads
72	95
96	60
108	23
120	1
144	6
162	5
192	2
216	22
288	37
324	4
360	3
384	10
432	2
576	1
648	3

$\|Aut\|$	spreads
864	13
1080	2
1152	12
1296	1
1440	1
3456	7
4320	1
4608	1
5184	1
13824	2
20736	1
41472	1
69120	1
All	72292

Table 4: $(7, 2, 5, 3)_1$ book spreads with all pages of type 3

$\|Aut\|$	spreads	$\|Aut\|$	spreads	$\|Aut\|$	spreads
1	6769	96	297	1920	1
2	395	128	61	2304	13
3	153	144	72	3072	3
4	25771	192	2531	3840	2
6	34	256	7	4608	6
8	1387	288	14	5760	1
12	1267	320	1	6144	2
16	17811	384	365	9216	1
24	249	512	3	11520	1
32	1259	576	51	18432	2
36	16	768	66	27648	1
48	3115	960	1	55296	1
64	176	1152	39	73728	1
72	14	1536	24	184320	1
				All	61984

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Table 5: $(7,2,5,3)_1$ book spreads with all pages of type 9

$\|Aut\|$	spreads	$\|Aut\|$	spreads	$\|Aut\|$	spreads
1	1	96	8	1152	1
2	2	128	4	1536	3
4	5	144	1	2048	1
8	4	192	5	3072	2
12	2	256	2	3840	1
16	2	384	6	4608	1
24	2	512	2	6144	1
32	4	768	2	11520	1
48	13	960	1	12288	1
64	6	1024	2	18432	1
				All	87

Table 6: $(7,2,5,3)_2$ book spreads with all pages of type 9

$\|Aut\|$	spreads	$\|Aut\|$	spreads	$\|Aut\|$	spreads
1	86	9	1	36	2
2	26	12	7	72	2
3	17	16	4	80	1
4	19	18	2	96	2
6	14	24	5	240	1
8	15	32	6	360	1
				All	211

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