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Performance of binary polar codes with high-dimensional kernel [1]

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Abstract. Upper union bound for the decoding error probability of polar codes with arbitrary binary kernels under the successive cancelation decoding is derived. The proposed approach is based on the representation of polar codes as multilevel ones.

1 Introduction

Polar codes is the first class of error correcting codes achieving the symmetric capacity of arbitrary binary-input discrete output-symmetric memoryless channel. It was shown in [6] that high-dimensional kernels (e.g. based on BCH codes) provide higher polarization rate than the Arikan kernel. That is, the decoding error probability of such polar codes decreases much faster with code length compared to similar Arikan codes. However, there are still no practical techniques besides simulations for estimating it.

In this paper, a novel method for computing an upper bound on decoding error probability of binary polar codes with high-dimensional kernels is introduced. It is based on representation of polar codes as a multilevel ones and exploits the techniques developed for their analysis. We consider only the case of two-layer polarizing transformation, but the results can be generalized to the case of arbitrary number of layers.

The paper is organized as follows. Section 2 introduces polar and multilevel codes. Section 3 presents the proposed error probability estimation method. Numeric results are given in Section 4. Finally some conclusions are drawn.

2 Background

2.1 Polar codes

A generator matrix of $(n = l^m, k)$ polar code consists of k rows of matrix $F^{\otimes m}$, where F is a $l \times l$ polarization kernel and $\otimes m$ denotes the m-times Kronecker product of a matrix with itself. Let u_i^m be a vector of length n that consist of k

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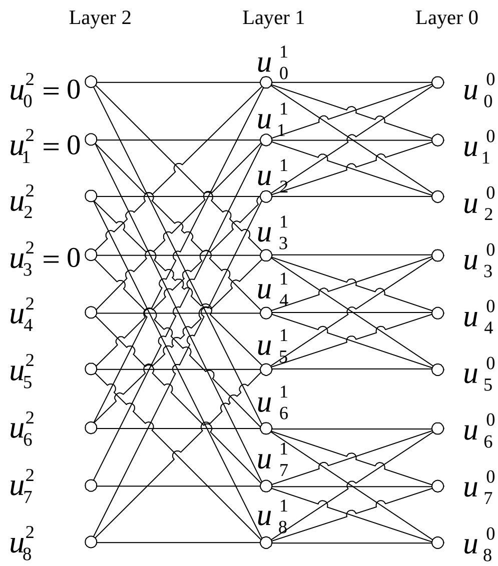

Figure 1: Encoding scheme for $(9, 6)$ polar code

information symbols and $n - k$ zero (frozen) symbols. Encoding operation can be represented as multiplication of u_i^m by $F^{\otimes m}$.

The polarizing transformation $F^{\otimes m}$ can be decomposed into $m + 1$ layers, which correspond to intermediate symbols u_i^j, $i = 0, \ldots, n - 1$, $j = 0, \ldots, m$ (see Fig. 1). Here layer 0 corresponds to codeword symbols, while layer m corresponds to information and frozen symbols.

For the sake of simplicity only the case of $m = 2$ will be considered below.

2.2 Multilevel codes

Multilevel coding is a coded modulation method, which is based on some labeling of a signal constellation $\mathcal{M}$, and a number of component codes $C^i, i = 0, \ldots, l - 1$, where l is the number of levels [1], [2]. A codewords of a multilevel code is a sequence of elements from $\mathcal{M}$, such that the i-th digits of the corresponding labels constitute a codeword of C^i.

$(n = l^2, k)$ polar code can be considered as a multilevel one [4]. Signal constellation is given by $\mathcal{M} = \{0, 1\}^l$. Labeling is given by

$$(u_{jl}^0, u_{jl+1}^0, \ldots, u_{jl+l-1}^0) = (u_{jl}^1, u_{jl+1}^1, \ldots, u_{jl+l-1}^1)F, \quad j = 0, \ldots, l - 1,$$

where $(u_{jl}^0, u_{jl+1}^0, \ldots, u_{jl+l-1}^0) \in \mathcal{M}$ is a constellation element to be transmitted, and $(u_{jl}^1, u_{jl+1}^1, \ldots, u_{jl+l-1}^1)$ is the corresponding label. Vector

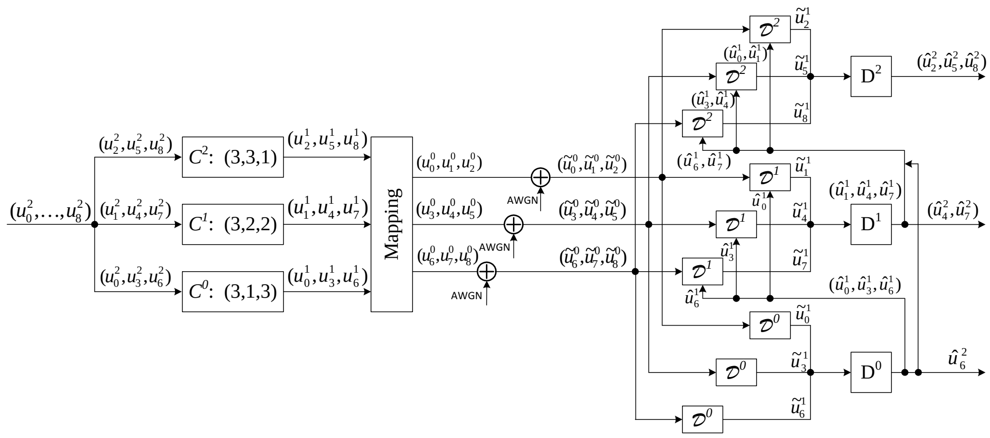

Figure 2: $(9, 6)$ polar code as a multilevel code

$(u_i^1, u_{i+l}^1, \ldots, u_{i+(l-1)l}^1)$ is obtained by encoding the payload data with C^i. In the case of polar codes C^i is also generated by some rows of kernel F.

After transmission of codeword $(u_0^0, u_1^0, \ldots, u_{l\cdot l-1}^0)$ over a binary-input memoryless output-symmetric channel one obtains LLRs $(\widetilde{u}_0^0, \widetilde{u}_1^0, \ldots, \widetilde{u}_{l\cdot l-1}^0)$. The multistage decoder proceeds by decoding the codewords of component codes at levels $0, 1, \ldots, l-1$. At the i-th level it computes the LLRs $\widetilde{u}_i^1, \widetilde{u}_{i+l}^1, \ldots, \widetilde{u}_{i+(l-1)l}^1$ under the assumption that estimates $\widehat{u}_{i'}^1, \widehat{u}_{i'+l}^1, \ldots, \widehat{u}_{i'+(l-1)l}^1$, $i' = 0, \ldots, i-1$ are correct. This corresponds to SISO decoding of a code $\overline{C}^i$ generated by last i rows of F. Let $\mathcal{D}^i$ represent a unit implementing this operation. The obtained LLRs are used by decoder D^i of component code C^i, which obtains codeword symbol estimates $\widehat{u}_i^1, \widehat{u}_{i+l}^1, \ldots, \widehat{u}_{i+(l-1)l}^1$, as well as the corresponding symbols of the payload data. Fig. 2 illustrates this construction.

3 Error probability

To obtain an expression for the decoding error probability observe that the multistage decoder makes an uncorrect decision if and only if any of component decoders make an error. The probability of correct decoding can be represented as

$$P\{e_0, \ldots, e_{l-1}\} = P\{e_{l-1}|e_0, \ldots, e_{l-2}\} P\{e_{l-2}|e_0, \ldots, e_{l-3}\} \ldots P\{e_1|e_0\} P\{e_0\},$$

where e_i denotes the event of correct decoding at the i-th level. Hence, the probability of error under the multistage decoding algorithm is given by

$$P = 1 - \prod_{i=0}^{l-1}(1 - p_i), \tag{1}$$

where $p_i = 1 - P\{e_i|e_0, \dots, e_{i-1}\}$ is the decoding error probability at level i.

Assume for the sake of simplicity that zero codeword was transmitted. Due to linearity of polar codes this does not affect the probability of error. Furthermore, it does not depend on the decisions at previous levels, provided that they are correct. At level i the decoder D^i makes decisions between subsets $\overline{C}^{i,0}$ and $\overline{C}^{i,1}$ of $\overline{C}^i \subset \mathcal{M}$, where code $\overline{C}^{i,s}$ contains all symbols with the i-th label digit equal to $s, s \in \{0, 1\}$. It can be seen that linear code $\overline{C}^i$ is partitioned into its subcode $\overline{C}^{i,0} = \overline{C}^{i+1}$, and a coset of this code $\overline{C}^{i,1}$.

An error occurs at the i-th level if there exist codewords $c^{(j)} = (c_0^{(j)}, \dots, c_{l-1}^{(j)}) \in \overline{C}^i, j = 0, \dots, l-1$, such that some of them belong to $\overline{C}^{i,1}$, and for any $\overline{c}^{(j)} \in \overline{C}^{i,0}$

$$L(c^{(0)}, c^{(1)}, \dots, c^{(l-1)}) > L(\overline{c}^{(0)}, \overline{c}^{(1)}, \dots, \overline{c}^{(l-1)}), \tag{2}$$

where $L(c) = \sum_{j=0}^{l-1}\sum_{q=0}^{l-1}\widetilde{u}^0_{jl+q}(2c_q^{(j)} - 1)$, $(v_0, v_1, \dots, v_{l-1}) \in C^i$, $v_j : c^{(j)} \in \overline{C}^{i,v_j}$. Since polar codes are linear, one can assume $\overline{c}^{(j)} = 0$.

This enables one to employ the techniques presented in [3] for estimation of the decoding error probability. The linearity of polar codes enables one to simplify the derivations.

Let $\overline{A}^{i,1}(Z) = \sum_{j=1}^{l}\overline{A}_j^{i,1} Z^j$ be the weight enumerator polynomial of $\overline{C}^{i,1}$. Each $v_j = 1$ can be mapped to any of $w = 2^{n-i-1}$ elements $c^{(j)} \in \overline{C}^{i,1}$ in (2). Let $\delta = \mathrm{wt}(v_0, \dots, v_{l-1})$. It can be seen that the total number of possible erroneous decisions of the decoder is w^δ. Let $s_t, t = 0, \dots, w-1$, be the number of times a particular element y_t of coset $\overline{C}^{i,1}$ appears in (2), so that $\sum_{t=0}^{w-1} s_t = \delta$. The total number of such configurations is given by $\frac{\delta!}{s_0!\dots s_{w-1}!}$. Multiplying this expression by $(Z^{\mathrm{wt}(y_0)})^{s_0}(Z^{\mathrm{wt}(y_1)})^{s_1}\dots(Z^{\mathrm{wt}(y_0)})^{s_{w-1}}$ and summing up over variables s_t, one obtains weight enumerator polynomial for the corresponding codewords of the multilevel code

$$\begin{aligned} S(Z) &= \sum_{\substack{s_0,\dots,s_{w-1} \\ s_0+\dots+s_{w-1}=\delta}} \frac{\delta!}{s_0!\dots s_{w-1}!}(Z^{\mathrm{wt}(y_0)})^{s_0}(Z^{\mathrm{wt}(y_1)})^{s_1}\dots(Z^{\mathrm{wt}(y_{w-1})})^{s_{w-1}} \\ &= (Z^{\mathrm{wt}(y_0)} + Z^{\mathrm{wt}(y_1)} + \dots + Z^{\mathrm{wt}(y_{w-1})})^\delta = (\overline{A}^{i,1}(Z))^\delta. \end{aligned} \tag{3}$$

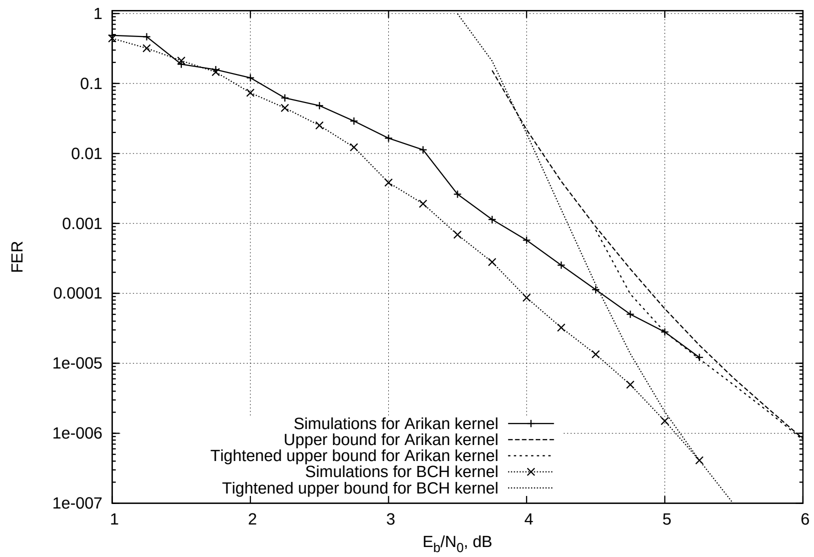

Figure 3: $(256, 128)$ polar codes

Let $A^i(Z)$ be the weight enumerator polynomial for code C^i. Then the weight enumerator polynomial for the multilevel code at level i is given by

$$\mathbb{A}^i(Z) = A^i(\overline{A}^{i,1}(Z)). \tag{4}$$

The coefficients of this polynomial can be used in the union bound for the decoding error probability at level i

$$p_i \leq \frac{1}{2} \sum_{j=1}^{l \cdot l} \mathbb{A}_j^i \operatorname{erfc}\left(\sqrt{j \frac{k}{n} \frac{E_b}{N_0}}\right). \tag{5}$$

The weight enumerator polynomial $\mathbb{A}^i(Z)$ describes all possible error events. However, these events are not independent. It was suggested in [5] to consider only Voronoi neighbours of zero codeword. Weight distribution for Voronoi neighbours of zero codeword for the case of binary linear block code is given by local weight profile of this code. The local weight profile $V^i(Z)$ for the code C^i is given by

$$V_j^i = |\left\{x \in C^i | \operatorname{wt}(x) = j, \nexists y \in C^i \setminus 0 : \operatorname{supp}(y) \subset \operatorname{supp}(x)\right\}|, j = 1, \ldots, l.$$

It can be defined in a similar way for the case of coset $\overline{C}^{i,1}$. Hence, one can replace $\overline{A}^{i,1}(Z)$ and $A^i(Z)$ in (4) with the corresponding local weight enumerator polynomials $\overline{V}^{i,1}(Z)$ and $V^i(Z)$, respectively.

4 Numeric results

Fig. 3 presents the performance of polar codes based on 16×16 Arikan kernel and 16×16 BCH kernel for the case of AWGN channel with BPSK modulation, as well as upper union bounds based on full weight spectrum of codes C^i and $\overline{C}^{i,1}$, and their local weight profile. It can be seen that both bounds coincide with the simulation results for high SNR values. Furthermore, employing local weight profiles in the case of Arikan kernel enables one to obtain tighter upper bound on the decoding error probability in the low-SNR region. In the case of BCH kernel upper bounds coincide, so only one of them is presented.

5 Conclusion

In this paper a novel method for computing an upper bound on decoding error pobability of binary polar codes with high-dimensional kernels was proposed. The proposed approach is based on representation of polar code as multilevel code and exploits the techniques developed in the area of multilevel coding.

References

[1] H. Imai and S. Hirakawa, “A new multilevel coding method using error correcting codes,” *IEEE Transactions on Information Theory*, vol. 23, no. 3, pp. 371–377, May 1977.

[2] U. Wachsmann, R. F. H. Fischer, and J. B. Huber, “Multilevel codes: Theoretical concepts and practical design rules,” *IEEE Transactions On Information Theory*, vol. 45, no. 5, pp. 1361–1391, July 1999.

[3] J. Huber, “Multilevel codes: Distance profiles and channel capacity,” in *ITG-Fachbericht 130, Conf. Rec.*, October 1994, pp. 305–319.

[4] P. Trifonov and P. Semenov, “Generalized concatenated codes based on polar codes,” in *Proceedings of IEEE International Symposium on Wireless Communication Systems*, 2011.

[5] E. Agrell, “Voronoi regions for binary linear block codes,” *IEEE Transactions on Information Theory*, vol. 42, no. 1, January 1996.

[6] S. B. Korada, E. Sasoglu, and R. Urbanke, “Polar codes: Characterization of exponent, bounds, and constructions,” *IEEE Transactions On Information Theory*, vol. 56, no. 12, December 2010.