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# Classification of the (12,19,1,2) and (12,20,1,2) superimposed codes [1]

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**Abstract.** In this paper the optimal superimposed codes with parameters $(12, 19, 1, 2)$ and $(12, 20, 1, 2)$ are classified up to equivalence. The values of $T(12, 1, 2)$ and $N(21, 1, 2)$ are obtained.

## 1 Introduction

**Definition 1.** *A binary $N \times T$ matrix $C$ is called an $(N, T, w, r)$ superimposed code (SIC) of length $N$ and size $T$ if for any pair of subsets $W, R \subset \{1, 2, ..., T\}$ such that $|W| = w$, $|R| = r$ and $W \cap R = \varnothing$, there exists a row $i \in \{1, 2, ..., N\}$ such that $c_{ij} = 1$ for all $j \in W$ and $c_{ij} = 0$ for all $j \in R$.*

The main problem in the theory of the superimposed codes is to optimize one of the parameters $N$ or $T$ for given values of the others. Two versions are considered:

- find the minimum length $N(T, w, r)$ for which an $(N, T, w, r)$ SIC exists;
- find the maximum size $T(N, w, r)$ for which an $(N, T, w, r)$ SIC exists.

The exact values of $N(T, 1, 2)$ are known for $T \leq 20$ ( [2–4]).

| $T$ | 3 | 4 | 5 | 6 | 7 | 8 | $9-12$ | 13 | $14-17$ | $18-20$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $N(T,1,2)$ | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |

**Definition 2.** *Two $(N, T, w, r)$ superimposed codes are equivalent if one of them can be transformed into the other by a permutation of the rows and a permutation of the columns.*

The number of nonequivalent classes of optimal $(N(T, 1, 2), T, 1, 2)$ superimposed codes for $T \leq 17$ is presented in [4]:

| $T$ | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| # | 1 | 1 | 1 | 1 | 2 | 4 | 25 | 4 | 1 | 1 | 5 | 2705 | 278 | 21 | 2 |

In this paper the optimal superimposed codes with parameters $(12, 19, 1, 2)$ and $(12, 20, 1, 2)$ are classified up to equivalence. The values of $T(12, 1, 2)$ and $N(21, 1, 2)$ are obtained. The results have been obtained using an exhaustive computer search for the generation of $(N, T, 1, 1)$ and $(N, T, 1, 2)$ superimposed codes and *Q-extension* ( [1]) for code equivalence testing.

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## 2 Preliminaries

The following propositions and definition have been used to generate superimposed codes.

**Theorem 3.** *(Sperner Theorem [5])* $T(N,1,1) = \binom{N}{\lfloor N/2 \rfloor}$.

**Definition 4.** *The residual code $Res(C, x = v)$ of a superimposed code $C$ with respect to value $v$ in column $x$ is a code obtained by taking all the rows in which $C$ has value $v$ in column $x$ and deleting the $x^{th}$ entry in the selected rows.*

We denote by $S_x$ the characteristic set of column $x$ and by $L_p$ the characteristic set of row $p$. The following two lemmas are obvious:

**Lemma 5.** *Let $C$ be an $(N, T, 1, 2)$ superimposed code and $x$ be a column of $C$. Then $Res(C, x = 0)$ is an $(N - |S_x|, T - 1, 1, 1)$ superimposed code.*

**Lemma 6.** *Let $C$ be an $(N, T, w, r)$ superimposed code and $x$ be a column of $C$. The matrix $C' = C \backslash \{x\}$ is an $(N, T - 1, w, r)$ superimposed code.*

The following lemma is a relation between the weights of columns and those of rows. We refer to [3] for a proof.

**Lemma 7.** *Suppose $C$ is an $(N, T, 1, 2)$ superimposed code and $x$ is a column such that $|S_x| \leq 2$. Then there exists a row $p$ for which $c_{px} = 1$ and $|L_p| = 1$.*

**Lemma 8.** *Suppose $C$ is an $(N, T, 1, 2)$ superimposed code and $p$ is a row such that $|L_p| = 1$. Then there exists an $(N - 1, T - 1, 1, 2)$ superimposed code.*

*Proof.* In the matrix $C$ there exists a column $x$ for which $c_{px} = 1$. If we delete the column $x$ and the row $p$ of $C$, we will obtain an $(N - 1, T - 1, 1, 2)$ superimposed code. □

The next lemma gives a relation between $N(T, 1, 2)$ and $N(T - 1, 1, 2)$.

**Lemma 9.** $N(T - 1, 1, 2) \leq N(T, 1, 2) \leq N(T - 1, 1, 2) + 1$.

*Proof.* From Lemma 6, it follows that $N(T - 1, 1, 2) \leq N(T, 1, 2)$. Let $C$ be an $(N - 1, T - 1, 1, 2)$ superimposed code. The following matrix is an $(N, T, 1, 2)$ SIC:

$$\left(\begin{array}{c|c} 0 & \\ \vdots & C \\ 0 & \\ \hline 1 & 0\,0\ldots0\,0 \end{array}\right)$$

Therefore $N(T, 1, 2) \leq N(T - 1, 1, 2) + 1$. □

# 3 Classification of the (12,19,1,2) superimposed codes

The following propositions have been used to generate all inequivalent $(12, 19, 1, 2)$ superimposed codes :

**Lemma 10.** *Suppose $C$ is a $(12, 19, 1, 2)$ superimposed code and $x$ and $y$ are two different columns of $C$. Then $|S_x \cap \overline{S_y}| \geq 2$.*

*Proof.* The matrix $C$ is $(12, 19, 1, 2)$ superimposed code. Therefore $|S_x \cap \overline{S_y}| \geq 1$. If $|S_x \cap \overline{S_y}| = 1$ then there is a row $p$ of $C$ for which $|L_p| = 1$. According to Lemma 8 there exists an $(11, 18, 1, 2)$ superimposed code, which is a contradiction. □

**Lemma 11.** *Suppose $C$ is a $(12, 19, 1, 2)$ superimposed code and $x$ is a column of $C$. Then $3 \leq |S_x| \leq 6$.*

*Proof.* It is known that there is no $(11, 18, 1, 2)$ superimposed code. According to Lemmas 7 and 8 it follows that $|S_x| \geq 3$.

The residual code $Res(C, x = 0)$ is an $(N_1, 18, 1, 1)$ superimposed code. According to Sperner Theorem $N(18, 1, 1) = 6$, so $|S_x| \leq 6$. □

**Lemma 12.** *Let $C$ be a $(12, 19, 1, 2)$ superimposed code. Then there is no column $x$ of $C$ for which $|S_x| = 6$.*

*Proof.* Suppose $x$ is a column of $C$ for which $|S_x| = 6$. We may assume that $x$ is the first column of $C$. So the matrix $C$ is of the form

$$\left(\begin{array}{c|c} 0 & \\ \vdots & C_0 \\ 0 & \\ \hline 1 & \\ \vdots & X \\ 1 & \end{array}\right)$$

where $C_0$ is $(6, 18, 1, 1)$ SIC. We may assume that:
– the rows and the columns of the matrix $C_0$ are sorted lexicographically;
– the rows of the matrix $X$ are sorted lexicographically;
– all columns of $C$ have weight between 3 and 6;
– $|S_y \cap \overline{S_z}| \geq 2$ for every two columns $y$ and $z$ of $C$.

Using author's computer program and the program *Q-extension* for code equivalence testing, we obtain that there is exactly 3 inequivalent possibilities for $C_0$. Using an exhaustive computer search, we tried to construct the matrix $X$. It turned out that the extension is impossible. □

**Lemma 13.** *Let $C$ be a $(12, 19, 1, 2)$ superimposed code. Then there is a row $p$ of $C$ for which $|L_p| \leq 7$.*

*Proof.* Suppose there is no row $p$ of $C$ for which $|L_p| \leq 7$. Therefore $\sum_{x \in C} |S_x| \geq 8.12 = 96$ and there is column of $C$ which have weight 6, a contradiction. □

**Theorem 14.** *There are exactly* 594 *inequivalent* $(12, 19, 1, 2)$ *superimposed codes.*

*Proof.* According to Lemma 13 there is a row $p$ of $C$ for which $|L_p| \leq 7$. We may assume that the row $p$ is the first row of $C$. So the matrix $C$ is of the form

$$\left(\begin{array}{c|ccccccc} 0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 & c_{1,13} & c_{1,14} & c_{1,15} & c_{1,16} & c_{1,17} & c_{1,18} & c_{1,19} \\ \hline & & & & & & & \\ A & & & & & & & \\ & & & & & & & \end{array}\right)$$

where the matrix $A$ is an $(11, 12, 1, 2)$ superimposed code. We may assume that:
– the rows and the columns of $A$ are sorted lexicographically;
– the last 7 columns of $C$ are sorted lexicographically;
– all columns of $C$ have weight between 3 and 5;
– $|S_y \cap \overline{S_z}| \geq 2$ for every two columns $y$ and $z$ of $C$.

We construct the matrix $A$ column by column, using author's $(N, T, 1, 2)$ superimposed code generation program and *Q-extension* for code equivalence testing. For the beginning, we found all possibilities for the first 5 columns, which form an $(11, 5, 1, 2)$ superimposed code. We obtain 327325 superimposed codes. Using *Q-extension* we find that there are exactly 15119 inequivalent superimposed codes among them. Then we extend each of these codes by appending one column to be an $(11, 6, 1, 2)$ SIC. Using *Q-extension* we find that there are over 160000 inequivalent possibilities for the first 6 columns. Similarly we make an extension to an $(11, 7, 1, 2)$ SIC. Thus we find that there are exactly 1012512 possibilities for the first 7 columns of the matrix $A$. We extend each of these codes to an $(11, 12, 1, 2)$ SIC. Using *Q-extension* we obtain 239232 inequivalent $(11, 12, 1, 2)$ possibilities for $A$.

Then we extend each of these matrices to $(12, 19, 1, 2)$ superimposed code. Finally we obtain that there are exactly 594 inequivalent $(12, 19, 1, 2)$ superimposed codes. □

# 4 Classification of the (12,20,1,2) superimposed codes

Using the method described, we extend each inequivalent $(12, 19, 1, 2)$ superimposed code by appending one column to be a $(12, 20, 1, 2)$ SIC. In this way we prove the following theorem:

**Theorem 15.** *There are exactly* 5 *inequivalent* $(12, 20, 1, 2)$ *superimposed codes.*

We tried to extend each $(12, 20, 1, 2)$ SIC by appending one column to be a $(12, 21, 1, 2)$ SIC. We found out that there is no $(12, 21, 1, 2)$ SIC. Therefore:

**Theorem 16.** $T(12, 1, 2) = 20$.

Using Lemma 9 we prove the following theorem:

**Theorem 17.** $N(21, 1, 2) = 13$.

# 5 Appendix

The representatives of all inequivalent (12, 20, 1, 2) superimposed codes

| **1** | **2** | **3** |
|---|---|---|
| 00000000000000011111 | 00000000000000011111 | 00000000000000011111 |
| 00000000000111100001 | 00000000000111100001 | 00000000000111100001 |
| 00000000111000100010 | 00000000111000100010 | 00000000111000100010 |
| 00000011001001000100 | 00000011001001000100 | 00000011001001000100 |
| 00000101010010001000 | 00000101010010001000 | 00000101010010001000 |
| 00001001100100010000 | 00001001100100010000 | 00001001100100010000 |
| 00110010000000101000 | 00110010000000110000 | 00110010000000110000 |
| 01010100000001010000 | 01010100000001000010 | 01010100000001000010 |
| 01101000000010000010 | 01101000000010000100 | 01101000001010000000 |
| 10011000010000000100 | 10011000010000000001 | 10011000010000000001 |
| 10100100001100000000 | 10100100001100000000 | 10100100000100000100 |
| 11000010100000000001 | 11000010100000001000 | 11000010100000001000 |

| **4** | **5** |
|---|---|
| 00000000000000011111 | 00000000000000011111 |
| 00000000000111100001 | 00000000000111100001 |
| 00000000111000100010 | 00000000111000100010 |
| 00000011001001000100 | 00000011001001000100 |
| 00000101010010001000 | 00000101010010001000 |
| 00001001100100010000 | 00001001100100010000 |
| 00110010000010000010 | 00110010000010000010 |
| 01010100000000110000 | 01010100000000110000 |
| 01101000001000000001 | 01101000001000001000 |
| 10011000010001000000 | 10011000010001000000 |
| 10100100000100000100 | 10100100000100000100 |
| 11000010100000001000 | 11000010100000000001 |

# References


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[3] H.K. Kim and V. Lebedev, On optimal superimposed codes, *J. Combin. Designs*, 12, 2004, 79–91.

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[5] E. Sperner, Ein Satz über Untermengen einer endlichen Menge, *Mathematische Zeitschrift*, 27, 1928, 544–548.