Thirteenth International Workshop on Algebraic and Combinatorial Coding Theory
June 15-21, 2012, Pomorie, Bulgaria *pp. 228–233*

(xv_t, xv_{t-1})-minihypers in $\mathrm{PG}(t, q)$ [1]

I. Landjev ivan@math.bas.bg
New Bulgarian University, 21 Montevideo str., 1618 Sofia, Bulgaria
P. Vandendriessche pv@cage.ugent.be
Ghent University, Department of Mathematics,
Krijgslaan 281 - Building S22, 9000 Ghent, Belgium

Abstract. We study (xv_t, xv_{t-1})-minihypers in $\mathrm{PG}(t, q)$, i.e. minihypers with the same parameters as a weighted sum of x hyperplanes. We classify these minihypers as a nonnegative rational sum of hyperplanes and we use this classification to extend and improve the main results of several papers which have appeared on the special case $t = 2$. We establish a new link with geometric coding theory and we use this link to create new families of these minihypers, which results in new families of linear codes meeting the Griesmer bound.

1 Introduction and preliminaries

Definition 1. *Let $\mathbb{F}_q$ be a finite field of order q and let G be an $m \times n$ matrix of rank k over $\mathbb{F}_q$. The* linear $[n, k]$-code C *defined by G is the k-dimensional subspace of $\mathbb{F}_q^n$ generated by the rows of G. The matrix G is called the* generator matrix *of C. The parameters n and k are respectively called the* length *and* dimension *of the code C and are denoted by $\mathrm{len}(C)$ and $\dim(C)$ respectively.*

Definition 2. *Let C be a linear code. If d is the maximum integer for which every two different vectors in C differ in at least d positions, then d is called the* minimum distance *of the code C.*

A linear $\mathbb{F}_q$-code with parameters n, k and d is denoted as an $[n, k, d]$-code.

Theorem 1 (The Griesmer Bound [3, 9])**.** *Let C be a linear $[n, k, d]$-code over $\mathbb{F}_q$. Then*

$$n \geq \sum_{i=0}^{k-1} \left\lceil \frac{d}{q^i} \right\rceil . \tag{1}$$

An important problem in coding theory is the study of linear codes that meet the Griesmer bound, as these have the shortest possible length for given dimension and minimum distance.

[1] The second author is supported by a PhD fellowship of the Research Foundation - Flanders (FWO).

Remark 1. *The projective* t*-dimensional space over the field* $\mathbb{F}_q$ *is denoted by* $\mathrm{PG}(t,q)$*. The number of points in this space is* $v_{t+1} = \frac{q^{t+1}-1}{q-1}$*. There are* v_{t+1} *hyperplanes in* $\mathrm{PG}(t,q)$*, which we will denote by* $H_1, \ldots, H_{v_{t+1}}$*.*

Definition 3. *An* (f,m)*-minihyper in* $\mathrm{PG}(t,q)$ *is a multiset of* f *projective points, with the property that every hyperplane contains at least* m *of these points. Hereby, the number of points in a multiset is always counted by summing the multiplicity of all points in the space or in a hyperplane.*

In case the parameters t and q are clear from the context (or not relevant), we will simply call such a multiset an (f,m)*-minihyper*. Clearly, the sum (as multiset) of a collection of subspaces $\{\pi_i\}_{i\in I}$ of the projective space, is an example of a $(\sum_{i\in I} v_{\dim(\pi_i)+1}, \sum_{i\in I} v_{\dim(\pi_i)})$-minihyper.

In [4], it was shown that the existence and classification of $[n,k,d]$-codes over $\mathbb{F}_q$ meeting the Griesmer bound can in many cases be reduced to studying the existence and classification of certain corresponding families of minihypers in $\mathrm{PG}(k-1,q)$. Since then, many papers have appeared on the link between minihypers and linear codes meeting the Griesmer bound.

In [1,6,7], (xv_2, xv_1)-minihypers in $\mathrm{PG}(2,q)$ were extensively studied; these are minihypers with the same parameters as a sum of x projective lines. We will study a more general class of minihypers, containing the previous class: (xv_t, xv_{t-1})-minihypers in $\mathrm{PG}(t,q)$, for arbitrary $t \geq 2$. These are minihypers with the same parameters as the sum of x hyperplanes.

In Section 2, we will establish a new classification of these minihypers in terms of rational linear combinations of incidence vectors of hyperplanes. In Section 3, we utilize this classification to extend and improve several key results from [6] and [7]. In Section 4, we establish a new link with coding theory, more specifically with the codewords in the $\mathbb{Z}_q$-ring code with the incidence matrix of points and hyperplanes in $\mathrm{PG}(n,q)$ as its parity check matrix. Using this link, we provide a new construction technique for these minihypers, resulting in several new constructions.

2 Rational sums

Definition 4. *A* proper multiset *in* $\mathrm{PG}(t,q)$ *is a nonempty multiset in which not all the points of* $\mathrm{PG}(t,q)$ *have positive multiplicity (so at least one point has zero multiplicity). A* proper minihyper *is a minihyper which is a proper multiset. Any non-proper* (f,m)*-minihyper contains the entire projective space; it is either equal to the projective space (and then it is a* (v_{t+1}, v_t)*-minihyper), or it is a decomposable minihyper (as it can be written as the sum of a* (v_{t+1}, v_t)*-minihyper and a* $(f - v_{t+1}, m - v_t)$*-minihyper).*

Clearly, an (xv_t, xv_{t-1})-minihyper is always proper for $x \leq q$, since there are only $xv_t \leq qv_t = v_{t+1} - 1 < v_{t+1}$ points in it.

Theorem 2. *Let $\mathfrak{K}$ be an arbitrary multiset in* $\mathrm{PG}(n,q)$*,* $q = p^h$*. Then its incidence vector* w *can uniquely be written as a linear combination of incidence vectors of hyperplanes over* $\mathbb{Q}$*:* $w = \sum_{i=1}^{v_{t+1}} r_i \chi_{H_i}$*. Moreover,* $r_i \geq 0$ *for each* $i \in \{1, \ldots, v_{t+1}\}$ *if and only if* w *is an* (f,m)*-minihyper with* $m \geq \frac{v_{t-1}}{v_t} f$*. If* $\mathfrak{K}$ *is proper,* $r_i \geq 0$ *for each* $i \in \{1, \ldots, v_{t+1}\}$ *if and only if* $\mathfrak{K}$ *is an* (xv_t, xv_{t-1})*-minihyper for some positive integer* x*.*

Theorem 3. *For any proper* (xv_t, xv_{t-1})*-minihyper* $\mathfrak{F} = \sum_{i=1}^{v_{t+1}} r_i \chi_{H_i}$ *in* $\mathrm{PG}(t,q)$*, the smallest positive integer* c *for which* $cr_i \in \mathbb{N}$ *for all* $i \in \{1, \ldots, v_{t+1}\}$*, is a power of* p *(and a divisor of* q^{t-1}*).*

Remark 2. *From now on, if* $\mathfrak{F}$ *is an* (xv_t, xv_{t-1})*-minihyper in* $\mathrm{PG}(t,q)$*, we will denote by* $r_i(\mathfrak{F})$ *the coefficient* r_i *associated to the* i*-th hyperplane* H_i *in the rational sum obtained in Theorem 2. If* π *is the hyperplane* H_i*, we may also write* $r_\pi(\mathfrak{F})$*. If the minihyper* $\mathfrak{F}$ *is clear from the context, we will simply write* r_i *or* r_π*. Since the minihyper can be written as a rational sum in a unique way, this will often be the case. In a similar fashion, we will write* $c(\mathfrak{F})$ *for the integer* c *from Theorem 3. Again, if the minihyper* $\mathfrak{F}$ *is clear from the context, we will simply write* c*.*

Remark 3. *A proper* (xv_t, xv_{t-1})*-minihyper in* $\mathrm{PG}(t,q)$ *(with* $x > 0$*) cannot be decomposed into a hyperplane and an* $((x-1)v_t, (x-1)v_{t-1})$*-minihyper if and only if* $r_\pi < 1$ *for each hyperplane* π*. In this case, we call the minihyper* hyperplane-indecomposable*. For* $x \leq q$*, we will see in Section 3 that hyperplane-indecomposability is equivalent to indecomposability.*

3 Extension of previous results

In this section, we will apply Theorem 2 to generalize and improve several key results from [6] and [7]. In what follows, we let $q = p^h$ with p prime (this defines p and h).

R. Hill and H.N. Ward [6] proved the following modular result via polynomial techniques for $t = 2$. This was extended to $t > 2$ in [5, Theorem 4.6], using similar techniques.

Theorem 4. *Let* $\mathfrak{F}$ *be an* (xv_t, xv_{t-1})*-minihyper with* $x \leq q - p^f$ *in* $\mathrm{PG}(t,q)$*, for some nonnegative integer* f*. Then* $\mathfrak{F}(\pi) \equiv xv_{t-1} \pmod{p^{f+1}q^{t-2}}$ *for every hyperplane* π *in* $\mathrm{PG}(t,q)$*.*

We managed to prove a sharper result:

Theorem 5. *Let* $\mathfrak{F}$ *be an* (xv_t, xv_{t-1})*-minihyper in* $\mathrm{PG}(t,q)$*. Then* $\mathfrak{F}(\pi) \equiv xv_{t-1} \pmod{\frac{q^{t-1}}{c}}$ *for every hyperplane* π *in* $\mathrm{PG}(t,q)$*. This is stronger than Theorem 4 since* p^{f+1} *divides* $\frac{q}{c}$*.*

Corollary 1. *Let $\mathfrak{F}$ be a nonempty (xv_t, xv_{t-1})-minihyper in* $\mathrm{PG}(t, q)$*. Then $x > q - \frac{q}{c}$. In other words: if $x \leq q - \frac{q}{c_0}$ for some positive integer c_0, then $c < c_0$.*

As special case of Corollary 1, we get the following corollary.

Corollary 2. *For $x \leq q - \frac{q}{p}$ (and hence for $x < q$ when $q = p$), we have $c = 1$. This means that the minihyper consists of a sum of x hyperplanes.*

This special case was proven earlier for $t = 2$ in [6, Theorem 20] and for general t in [5, Corollary 4.8]. The sharpness of the bound in Corollary 2 has not yet been demonstrated. In Section 4, we will show the sharpness of this bound. This family of examples will show the sharpness of the bound in Corollary 1 in general when $c = p^e$ with $e|h$ (with $q = p^h$).

Corollary 3. *If $x \leq 2q - 2\frac{q}{p} + 1$, then a proper (xv_t, xv_{t-1})-minihyper is decomposable if and only if it is hyperplane-decomposable.*

Remark 4. *Corollary 2 and its sharpness determine the smallest x for which there is an indecomposable (or, equivalently, a hyperplane-indecomposable) (xv_t, xv_{t-1})-minihyper in* $\mathrm{PG}(t, q)$.

The largest x for which there exists a hyperplane-indecomposable (xv_t, xv_{t-1})-minihyper is $x = q^t - q$, in which case $\mathfrak{F}$ is $q^{t-1} - 1$ times the setwise complement of the unique point with multiplicity 0*. The largest x for which a proper indecomposable minihyper exists is not known, not even for t=2. A generalization of the result by Landjev and Storme [7] on the case $t = 2$ follows straightforwardly from the techniques in this paper; it is presented in Theorem 6. We however believe that this bound is not sharp at all.*

Theorem 6. *Let $\mathfrak{F}$ be a hyperplane-indecomposable (xv_t, xv_{t-1})-minihyper. Then $x \leq q^t - 2q + \frac{q}{p} - 1$ and the multiplicity of any point in $\mathfrak{F}$ is at most $q^{t-1} - 1$.*

4 Another link with coding theory

We have established a new correspondence between the hyperplane-indecomposable (xv_t, xv_{t-1})-minihypers in $\mathrm{PG}(t, q)$ and the dual projective space code of $\mathrm{PG}(t, q)$ over the ring $\mathbb{Z}_c$, where c is the number described in Theorem 3.

Theorem 7. *There is a natural bijective correspondence between the code words*

$$(z_1, \ldots, z_{v_{t+1}}) \in C_c^{\perp}(t, q)$$

and the hyperplane-indecomposable (xv_t, xv_{t-1})-minihypers $\sum_{i=1}^{v_{t+1}} r_i \pi_i$ (with c the number from Theorem 3).

Using this correspondence, some new constructions of non-trivial (xv_t, xv_{t-1})-minihypers can be done. Ball's construction, mentioned in [7], can be derived as a special case of this theorem.

Lemma 1 (Ball's construction)**.** *Let B be a set of points in $\mathrm{PG}(t,q)$ and let e be the largest nonnegative integer such that B meets each hyperplane in 0 modulo p^e points. Then there exists an $\left(\frac{|B|}{p^e}v_t, \frac{|B|}{p^e}v_{t-1}\right)$-minihyper in $\mathrm{PG}(t,q)$ with $c = p^e$.*

More interestingly, we can also utilize 1 modulo p^e sets to construct new examples, as the following lemma demonstrates.

Lemma 2. *Let A and B be sets of points in $\mathrm{PG}(t,q)$ and let e be the largest nonnegative integer such that A and B both meet each hyperplane in 1 modulo p^e points. Then there exists an (xv_t, xv_{t-1})-minihyper $\mathfrak{F}$ in $\mathrm{PG}(t,q)$ with $c = p^e$ and $x = |B \setminus A| + \lambda\frac{|A|-|B|}{p^e}$, for any $\lambda \in \{1, 2, \ldots, p^e - 1\}$.*

Several examples of 1 modulo p^e sets (with $e \geq 1$) are known: i-dimensional subspaces with $i \geq 1$, Baer subgeometries, unitals and hermitian varieties, linear blocking sets and many, many other commonly studied structures in finite geometries. With Lemma 2, all of them can be used to obtain structurally new examples. In particular, we were able to construct a minimal nontrivial example, i.e. a minihyper with $x = q - \frac{q}{p} + 1$ which is not a sum of hyperplanes. This shows the sharpness of Corollary 2 and can also be used to show the sharpness of Theorem 6. In some cases, the construction can also be used to show the sharpness of Corollary 1.

Theorem 8. *For each divisor e of h (where $q = p^h$), there exists an (xv_t, xv_{t-1})-minihyper in $\mathrm{PG}(t,q)$ with $x = q - \frac{q}{p^e} + 1$.*

Remark 5. *Let again $t = 2$ and let $q = p^2$ and $e = 1$. Repeating the construction in the proof of Theorem 8 with the same choices of A and B, but now varying $\lambda \in \{1, \ldots, p-1\}$, one obtains a spectrum result: a nontrivial minihyper for each $x \in \{q - \frac{q}{p} + 1, \ldots, q-1\}$.*

The construction in the proof of Theorem 8 was inspired by the construction of the smallest known code words (in terms of Hamming weight) in the dual code $C^{\perp}_{\mathrm{PG}(2,q)}$ associated to the projective plane $\mathrm{PG}(2,q)$ [8]. These code words are conjectured to be the smallest in Hamming weight. Corollary 2 shows that they are the smallest weight code words with respect to the modified weight function $w : C^{\perp}_{\mathrm{PG}(2,q)} \to \mathbb{N} : (c_1, \ldots, c_{v_{t+1}}) \mapsto \sum_{i=1}^{v_{t+1}} c_i$.

Corollary 4. *The bound in Corollary 2 is sharp. When e divides h (with $c = p^e$ and $q = p^h$), the bound in Corollary 1 is also sharp.*

It is not known whether the bound in Corollary 1 is sharp for all c.

References

[1] S. Ball, R. Hill, I. Landjev and H. Ward, On $(q^2+q+2, q+2)$-arcs in the projective plane $\mathrm{PG}(2,q)$, *Des. Codes Cryptogr.* **24**, 205–224 (2001).

[2] P.J. Cameron and J.H. Van Lint, *Designs, Graphs, Codes and their Links*, Cambridge University Press, 1991.

[3] J.H. Griesmer, A bound for error-correcting codes, *IBM J. Res. Develop.* **4**, 532–542 (1960).

[4] N. Hamada, Characterization of minihypers in a finite projective geometry and its applications to error-correcting codes, *Bull. Osaka Women's Univ.* **24**, 1–24 (1987).

[5] P. Herdt, $[n,k,d]_q$-Codes mit $k \geq 3$, $d = rq^{k-2}$ und $n = \lceil r/q \rceil + r + rq + \cdots + rq^{k-2}$, $r \in \mathbb{N}$, Msc. Thesis at Justus-Liebig-Universität Gießen.

[6] R. Hill and H. Ward, A geometric approach to classifying Griesmer codes, *Des. Codes Cryptogr.* **44**, 169–196 (2007).

[7] I.N. Landjev and L. Storme, A study of $(x(q+1), x; 2, q)$-minihypers, *Des. Codes Cryptogr.* **54**, 135–147 (2010).

[8] M. Lavrauw, L. Storme and G. Van de Voorde, On the code generated by the incidence matrix of points and hyperplanes in $\mathrm{PG}(n,q)$ and its dual, *Des. Codes Cryptogr.* **48**, 231–245 (2008).

[9] G. Solomon and J.J. Stiffler, Algebraically punctured cyclic codes, *Inform. and Control* **8**, 170–179 (1965).