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On the sharpness of the Jamison-Bruen bound

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Abstract. In this note we investigate the sharpness of the Jamison-Bruen bound for t-fold blocking sets with respect to hyperplanes in $\mathrm{AG}(n, q)$.

1 The Jamison-Bruen bound

A t-fold blocking set with respect to hyperplanes in $\mathrm{AG}(n, q)$ is a set B of points such that each hyperpalne intersects it in at least t points. A t-fold blocking set of cardinality N is also referred to as an (N, t)-blocking set. The problem of finding the minimal size of an affine blocking set with respect to hyperplanes in $\mathrm{AG}(n, q)$ is interesting in its own right, but it has also a nice application to coding theory due to the following theorem.

Theorem 1. *The existence of the following objects is equivalent:*

(1) *an* $[n, k, d]_q$ *linear code with a word of maximal weight* n*;*

(2) *an* $(n, n - d)$*-arc in* $\mathrm{PG}(k - 1, q)$ *with an empty hyperplane;*

(3) *an affine* $(q^{k-1} - n, q^{k-2} - n + d)$*-blocking set in* $\mathrm{AG}(k - 1, q)$.

In [4] Jamison and later on Brouwer and Schrijver [2] proved that for an 1-fold blocking set B in $\mathrm{AG}(n, q)$

$$|B| \geq n(q - 1) + 1. \tag{1}$$

This bound was generalized by Bruen in [3]. He proved that if B is a t-fold blocking set with respect to hyperplanes in $\mathrm{AG}(n, q)$ then

$$|B| \geq (t + n - 1)(q - 1) + 1. \tag{2}$$

It was pointed out in [2] that in case of $s = 1$ equality is achieved in all affine geometies $\mathrm{AG}(n, q)$. To check this just observe that a blocking set in $\mathrm{AG}(n, q)$ can be obtained as a $((n + t - 2)(q - 1) + 1, 1)$-blocking set in a hyperplane H (isomorphic to $\mathrm{AG}(n - 1, q)$) plus $q - 1$ additional points – one point for each of the $q - 1$ hyperplanes parallel to H.

2 The improvement by Ball

Ball [1] has improved on Bruen's result by proving the following theorem.

Theorem 2. *For $t < q$ a t-fold blocking set with respect to hyperplanes in* $\mathrm{AG}(n, q)$ *has at least $(t + n - 1)(q - 1) + k$ points provided there exists a j such that $k - 1 \leq j < t$ and the binomial coefficient*

$$\binom{k-n-t}{j} \not\equiv 0 \pmod{p}.$$

Setting $k = t$ and $j = k - 1$ in Theorem 2, one gets

Corollary 3. *For $t < q$ a t-fold blocking set with respect to hyperplanes in* $\mathrm{AG}(n, q)$ *has at least $(t + n - 1)q - n + 1$ points provided*

$$\binom{-n}{t-1} \not\equiv 0 \pmod{p}.$$

Corollary 4. *For $t < q$ a t-fold blocking set in* $\mathrm{AG}(2, q)$ *has at least $(t+1)q - p^e$ where e is maximal such that p^e divides t.*

3 Zanella's result

It turns out that for fixed t and prime power q, t-fold blocking sets with large t do not exist. Zanella proved in [5] that the bound (2) can be attained only for values of t with

$$t \leq \frac{1}{2}(n-1)(q-1) + 1.$$

Below we include a proof of a somewhat weaker version of Zanella's result based on the Griesmer bound.

Theorem 5. *There exists no t-fold blocking set in* $\mathrm{AG}(n, q)$ *meeting the Jamison-Bruen bound for every*

$$t > \frac{q^n}{(q^n-1)}(q-1)(n-1) + 1.$$

Proof. Assume B is an $((n + t - 1)(q - 1) + 1, t)$-blocking set in $\mathrm{AG}(n, q)$ with $t > \frac{q^n}{(q^n-1)}(q-1)(n-1)+1$. Then B can be viewed as an $((n+t-1)(q-1)+1, w)$-arc in $\mathrm{PG}(n, q)$ with $w \leq (n - 1)(q - 1) + 1$. Such an arc is equivalent to an $[N, n + 1, d]_q$-code with $N = (n + t - 1)(q - 1) + 1$ and $d = N - w \leq t(q - 1)$. By the Griesmer bound, we have

$$N \geq t(q-1) + \lceil\frac{t(q-1)}{q}\rceil + \ldots + \lceil\frac{t(q-1)}{q^n}\rceil.$$

Now from $t > \frac{q^n}{(q^n-1)}(q-1)(n-1)+1$ we get

$$t(q-1)+\lceil\frac{t(q-1)}{q}\rceil+\ldots+\lceil\frac{t(q-1)}{q^n}\rceil \geq$$
$$t(q-1)+\frac{t(q^n-1)}{q^n} > (n+t-1)(q-1)+1 = N, \quad (3)$$

a contradiction. □

4 Blocking sets meeting the Jamison-Bruen bound

It turns out that equality in (2) can be achieved for $n = 3$, $t = q-1$ (cf. [1]).

Theorem 6. *There exists a $(q^2, q-1)$-blocking set in* $\mathrm{AG}(3,q)$ *for every prime power q.*

Proof. Let L and M be two skew lines in $\mathrm{PG}(3,q)$. Denote by $x_0,\ldots,x_q$ the points on L and by $y_0,\ldots,y_q$ the points incident with M. Set $L_i = \langle x_i, y_i\rangle$, $i = 0,1,\ldots,q$, and let $\pi_i = \langle L, y_i\rangle$, $i = 0,\ldots,q$, be the planes through L.

The set $B = \cup_{i=0}^{q} L_i$ is a $((q+1)^2, q+1)$ blocking set. Clearly $B \setminus \pi_0$ is a $(q^2, q-1)$-blocking set in $\mathrm{PG}(3,q) \setminus \pi_0 \cong \mathrm{AG}(3,q)$. □

Deleting points from this blocking set we get blocking sets that lie close to (2).

Corollary 7. *There exists a $(q^2 - t(t+1), q-t-1)$-blocking set in* $\mathrm{AG}(3,q)$ *for every prime power q and every $t = 1,\ldots,q-2$.*

Proof. Delete t points on each of the lines $L_1,\ldots,L_{t+1}$ from the proof of Theorem 6. □

The next result is better than Corollary 7 for large q.

Corollary 8. *There exists a $(q^2 - q, q-4)$-blocking set in* $\mathrm{AG}(3,q)$ *for every prime power q.*

Proof. Delete a $(q,3)$-arc (q points in general position) from the blocking set in Theorem 6. □

5 A blocking set attaining the Jamison-Bruen bound

The following result generalizes the construction of a $(q^2, q-1)$-blocking set in $\mathrm{AG}(3,q)$. It has been pointed out by Ball that in all cases with $t+n = q+2$, Theorem 2 does not improve on (2). In fact, in this case blocking sets meeting the Jamison-Bruen bound can be constructed.

Theorem 9. *There exists a* $(q^2, q-n+2)$*-blocking set in* $\mathrm{AG}(n,q)$ *for every prime power* q *and every* $3 \le n \le q+1$.

Proof. Let T be subspace of codimension 2 in $\Omega = \mathrm{PG}(n,q)$ and denote by $H_0, \ldots, H_q$ the hyperplanes through T in Ω. Fix $q+1$ points in T that are in general position[1], a normal rational curve, say. Let $x_0, \ldots, x_q$ be these $q+1$ points. In each of the hyperpalnes H_i select a line meeting T in x_i. Now the set

$$B = \cup_{i=1}^{q}(L_i \setminus x_i)$$

is a blocking set in $\Omega \setminus H_0 \cong \mathrm{AG}(n,q)$.

We have to check that every hyperplane different from H_0 is blocked at least $q-n+2$ times. For the hyperplanes through T this is obvious since all they contain q points from B. A hyperplane H that does not contain T meets in a subspace of codimension 3 contained in T. Such a subspace contains at most $n-1$ of the points x_i. Hence meets at least $q-n+2$ of the lines $L_1, \ldots, L_q$ in points different from $x_1, \ldots, x_q$. □

6 Other blocking sets of small cardinality

The folowing theorem produces sometimes blocking sets with cardinality close to the lower bound.

Theorem 10. *If there exists a* (N, w)*-arc in* $\mathrm{PG}(n-1,q)$ *then there exists a* $(qN, N-w)$*-blocking set in* $\mathrm{AG}(n,q)$.

Proof. Choose N lines $L_1, \ldots, L_N$ in $\mathrm{PG}(n,q)$ such that the set of points

$$\{P_i = L_i \cap H_\infty \mid i = 1, \ldots, N\}$$

forms an (N,w)-arc in the $(n-1)$-dimensional projective space at infinity H_∞. The set of points B on these lines in the affine space $\mathrm{AG}(n,q) = \mathrm{PG}(n,q) \setminus H_\infty$ is the desired blocking set. □

Another good constructions comes from the blocking sets constructed in Theorem 9.

Theorem 11. *For every* $s = 0, 1, \ldots, q-1-n$*, there exists an affine* $(q^2 - s(n-s+2), q-(n-2+s))$*-blocking set in* $\mathrm{AG}(n,q)$.

Proof. Start with a blocking set of the type described in Theorem 9. It consists of the points of q mutually skew lines in $\mathrm{AG}(n,q)$ meeting the hyperplane at infinity in a q-arc. Remove $n-2+s$ points from each of the lines $L_1, \ldots, L_s$. Now it easily checked that the obtained blocking set has the desired parameters. □

[1] no $u+2$ points lie in a subspace of dimension u

Let us note that for $s = 0$ and $s = q + 1 - n$, we get blocking sets meeting (2).

Ball proved that for double blocking sets the lower bound given by Theorem 2 is attained. The problem of finding the minimal size of a triple blocking set in AG(n, q) stays open. By Theorem 10, we get blocking sets of size $(n+2)q$, while the lower bound is $(n + 2)q - n - 1$ or $(n + 2)q - n + 1$ depending on whether $\binom{-n}{t-1}$ is 0 modulo the characteristic of the field. The table below gives the best known triple blocking sets for small fields and small dimensions. The lower bounds are obtained by (2) or Theorem 2.

Bounds for 3-fold blocking sets in AG(n, q)**,** $n = 3, 4, 5$**.**

	AG$(3, q)$			AG$(4, q)$			AG$(5, q)$		
q	LB	UB	Comment	LB	UB	Comment	LB	UB	Comment
4	16	16	Thm 6						
5	23	23	Cor 7	25	25	Thm 6			
7	33	35	Thm 10	39	41	Thm 11	45	45	Thm 11
8	36	40	Thm 10	44	48	Thm 10	52	54	Thm 11
9	41	45	Thm 10	51	54	Thm 10	57	63	Thm 10
11	53	55	Thm 10	63	66	Thm 10	73	77	Thm 10
13	63	65	Thm 10	75	78	Thm 10	87	91	Thm 10

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